CONVERGENCE ANALYSIS OF SYMMETRIC INTERPOLATORY SUBDIVISION SCHEMES

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Declaration

I, the undersigned, hereby declare that this thesis contains no material which I have previously in its entirety or part submitted at any other University for a degree, and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except by way of background information.

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Date

Summary

Subdivision schemes are simple iterative algorithms for generating curves and surfaces in computer-aided geometric design (CAGD). In applications it is often desired that the limit subdivision curve should interpolate the given control (or data) points. In this thesis we study the convergence of a class of symmetric interpolatory subdivision schemes with a given polynomial reproduction property.

In Chapter 1, we give a precise mathematical formulation of interpolatory subdivision schemes, together with the additional constraints of symmetry, polynomial reproduction property of degree $\leq 2m-1$, and locality linked to the integer $n \geq m$, yielding the corresponding class $\mathcal{A}_{m,n}$ of Laurent polynomial mask symbols. In addition, we introduce the concept of a corresponding interpolatory refinable function ϕ, and it is shown that, in this interpolatory case, the existence of ϕ implies the convergence of the corresponding interpolatory subdivision scheme.

Next, in Chapter 2, we use results from the literature to establish, in Theorem 2.6, a subdivision contractivity condition which guarantees the convergence of general (not necessarily interpolatory) subdivision schemes, and which is therefore directly applicable to $\mathcal{A}_{m,n}$.

Chapter 3 is devoted to the use of the cascade algorithm to develop a constructive existence result for interpolatory refinable functions corresponding to the Laurent polynomial mask symbol class $\mathcal{A}_{m,n}$. Our main result of this chapter, as given in Theorem 3.7, shows that subject to a further restrictive condition, a substantially weaker contractivity condition than the one in Theorem 2.6 guarantees subdivision convergence for this symmetric interpolatory case.

We proceed in Chapter 4 to use our Theorem 3.7 to establish, in Theorem 4.6, existence and convergence with respect to the optimally local mask symbol class $\mathcal{A}_{m,m}$, which contains only the Laurent polynomial $A = A^I_m$, and which is often referred to in the literature as Dubuc-Deslauriers subdivision. Whereas the corresponding interpolatory refinable function ϕ^I_m has in fact previously been shown to exist, by means of a proof based on Fourier transforms, our alternative approach based on cascade algorithm convergence makes the convergence analysis of this thesis self-contained. Moreover, our Theorem 4.6 enables us to remove the above-mentioned restrictive condition in Theorem 3.7, and hence to formulate Theorem 4.7 as our main result of this thesis, namely that, for Laurent polynomial mask symbols in $\mathcal{A}_{m,n}$, a substantially weaker contractivity condition than the one in Theorem 2.6, as applying to general (not necessarily interpolatory) subdivision, guarantees interpolatory refinable function existence and therefore also corresponding interpolatory subdivision convergence.

Finally, in Chapter 5, we apply our Theorem 4.7 to study the convergence of the one-parameter family $\mathcal{A}_{m,m+1}$ of symmetric interpolatory schemes, as fully characterized by the Laurent polynomial mask symbol family $A^I_m(t|\cdot) := (1-t)A^I_m + tA^I_{m+1}$, for any parameter value $t \in \mathbb{R}$. In particular, we show that, for $m \geq 2$, the $t-$interval for convergence thus obtained contains negative $t-$values not covered by previous convergence theories in the literature. In addition, we present Hölder regularity results for the associated interpolatory refinable function $\phi^I_m(t|\cdot)$. The results of this chapter is graphically illustrated for selected values of t inside its convergence interval.

Opsomming

Subdivisieskemas is eenvoudige iteratiewe algoritmes vir die generering van krommes en oppervlakke in geometriese ontwerp (CAGD). In toepassings word dit dikwels verlang dat die limiet subdivisiekromme die gegewe kontrolepunte (of datapunte) interpoleer. In hierdie tesis bestudeer ons 'n klas van simmetriese interpolerende subdivisieskemas met 'n gegewe polinoomreproduksie eienskap.

In Hoofstuk 1 gee ons 'n presiese wiskundige formulering van interpolerende subdivisieskemas, tesame met die addisionele beperkings van simmetrie, polinoomreproduksie van graad $\leq 2m-1$, en lokaliteit verbind aan die heelgetal $n \geq m$, wat dan die ooreenkomstige klas $\mathcal{A}_{m,n}$ van Laurent polinoom maskersimbole oplewer. Daarbenewens stel ons bekend die konsep van 'n ooreenkomstige interpolerende verfynbare funksie ϕ, en word daar aangetoon dat, vir hierdie interpolerende geval, die bestaan van ϕ die konvergensie van die ooreenstemmende interpolerende subdivisieskema impliseer.

Vervolgens, in Hoofstuk 2, gebruik ons bekende resultate vir die daarstelling in Stelling 2.6 van 'n subdivisie kontraktiwiteitsvoorwaarde wat die konvergensie van algemene (nie noodwendige interpolerende) subdivisieskemas waarborg, en wat dus direk toepasbaar is op $\mathcal{A}_{m,n}$.

Hoofstuk 3 word gewy aan die gebruik van die kaskade algoritme vir die ontwikkeling van 'n konstruktiewe bestaansresultaat vir interpolerende verfynbare funksies wat ooreenstem met die klas $\mathcal{A}_{m,n}$ van Laurent polinoom maskersimbole. Ons hoofresultaat van hierdie hoofstuk, soos gegee in Stelling 3.7, toon aan dat, onderworpe aan 'n verdere beperkende voorwaarde, 'n aansienlik swakker kontraktiwiteitsvoorwaarde as die een in Stelling 2.6, subdivisie konvergensie waarborg vir hierdie simmetriese interpolerende geval.

Ons gaan voort in Hoofstuk 4 om, in Stelling 4.6, en met behulp van Stelling 3.7, bestaan en konvergensie met betrekking tot die optimale lokale simboolklas $\mathcal{A}_{m,m}$ te bewerkstellig, waar $\mathcal{A}_{m,m}$ slegs die Laurent polinoom $A = A_m^I$, wat dikwels in die literatuur verwys word na as Dubuc-Deslauriers subdivisie, bevat. Waar die bestaan van die ooreenkomstige interpolerende verfynbare funksie ϕ_m^I vantevore bewys is met behulp van 'n bewys gebaseer op Fourier transforms, het ons alternatiewe benadering gebaseer op kaskade algoritme die gevolg dat die konvergensie analise van hierdie tesis selfbevattend is. Daarbenewens stel ons Stelling 4.6 ons in staat om die bogenoemde beperkende voorwaarde in Stelling 3.7 te verwyder, en dus om Stelling 4.7 as die hoofresultaat van hierdie tesis te formuleer, naamlik dat, vir Laurent polinoom maskersimbole in $\mathcal{A}_{m,n}$, 'n aansienlik swakker kontraktiwiteitsvoorwaarde as die een in Stelling 2.6, soos toepasbaar op algemene (nie noodwendig interpolerende) subdivisie, interpolerende verfynbare funksie bestaan, en dus ook ooreenstemmende interpolerende subdivisiekonvergensie, waarborg.

In die finale Hoofstuk 5 pas ons Stelling 4.7 toe om die konvergensie van die eenparameter klas $\mathcal{A}_{m,m+1}$ van simmetriese interpolasieskemas te bestudeer, soos ten volle gekarakteriseer deur die Laurent polinoom maskersimbool familie $A_m^I(t|\cdot) := (1-t)A_m^I + tA_{m+1}^I$, vir enige parameterwaarde $t \in \mathbb{R}$. In die besonder toon ons aan dat, vir $m \geq 2$, die t-interval vir konvergensie sodanig verkry, ook negatiewe t-waardes bevat wat nie gedek word deur vorige konvergensieresultate in die literatuur nie. Daarbenewens gee ons ook Hölder regulariteitsresultate vir die ooreenkomstige interpolerende verfynbare funksie $\phi_m^I(t|\cdot)$. Die resultate van hierdie hoofstuk word grafies geïllustreer vir geselekteerde waardes van t binne die konvergensie-interval daarvan.

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Contents

List of Symbols

Symbol	Definition
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^k$	the set $\{\mathbf{x} = (x_1, x_2, \ldots, x_k) : x_j \in \mathbb{R}, j = 1, \ldots, k\}$
$\sum_j$	the sum $\sum_{j\in\mathbb{Z}}$
$\mathbb{C}$	the set of complex numbers
$\lfloor x \rfloor$	the largest integer $\leq x$
$\lceil x \rceil$	the smallest integer $\geq x$
$M(\mathbb{Z})$	the linear space of bi-infinite real-valued sequences,i.e.$c \in M(\mathbb{Z}) \Longleftrightarrow c = \{c_j : j \in \mathbb{Z}\} \subset \mathbb{R}$
$M^s(\mathbb{Z})$	the linear space of bi-infinite real-valued sequences for $s = 1, 2$, or 3, i.e. $\mathbf{c} \in M^s(\mathbb{Z}) \Longleftrightarrow \mathbf{c} = \{\mathbf{c}_j : j \in \mathbb{Z}\} \subset \mathbb{R}^s$
$M_0(\mathbb{Z})$	the subspace of $M(\mathbb{Z})$ consisting of those sequences in $M(\mathbb{Z})$ with finite support, i.e. a sequence $c \in M(\mathbb{Z})$ is called *finitely-supported* if the set $\{j : c_j \neq 0, j \in \mathbb{Z}\}$ has a finite number of elements
$\mathbf{a} = \{a_j\}$	refinement mask, or subdivision mask in $M_0(\mathbb{Z})$
$\text{supp}(\mathbf{a})$	support of the mask $\mathbf{a}$, i.e. $\text{supp}(\mathbf{a}) = [\mu, \nu]\|_{\mathbb{Z}}$, where $\mu := \min\{j : a_j \neq 0\}$ and $\nu := \max\{j : a_j \neq 0\}$
A	$\sum_j a_j(\cdot)^j$, the subdivision mask symbol, a Laurent polynomial or a polynomial, corresponding to the mask $\{a_j\}$
$S_{\mathbf{a}}$	subdivision operator asociated with the mask $\mathbf{a} \in M_0(\mathbb{Z})$
$S_{\mathbf{a}}^r$	subdivision operator, with mask $\mathbf{a}$, applied r times
$\mathbf{c}^{(r)}$	the resulting sequence after applying $S_{\mathbf{a}}^r$ to a sequence $\mathbf{c} \in M^s(\mathbb{Z})$
$\Delta\mathbf{c}$	the backward difference sequence defined by $(\Delta\mathbf{c})_j = \Delta\mathbf{c}_j = \mathbf{c}_j - \mathbf{c}_{j-1}$, $j \in \mathbb{Z}$, if $\mathbf{c} \in M^s(\mathbb{Z})$
$\Delta^k\mathbf{c}$	the k-th backward difference sequence defined by $\Delta^m\mathbf{c} = \Delta(\Delta^{m-1}\mathbf{c})$, $m \geq 2$, if $\mathbf{c} \in M^s(\mathbb{Z})$
$\sup_j$	the supremum over all $j \in \mathbb{Z}$
$\sup_x$	the supremum over all $x \in \mathbb{R}$
$\|\mathbf{c}\|$	the Euclidean length $\sqrt{c_1^2 + \cdots + c_s^2}$ of a vector $\mathbf{c} = (c_1, \ldots, c_s) \in \mathbb{R}^s$, if $s \geq 2$.

Symbol	Definition
$\ell^\infty(\mathbb{Z})$	the subspace of bounded sequences in $M^s(\mathbb{Z})$,i.e. if $\mathbf{c} \in M^s(\mathbb{Z})$, and $\sup_j \lvert \mathbf{c}_j \rvert < \infty$
$C(\mathbb{R})$	the linear space of continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$
$C_0(\mathbb{R})$	the subspace of $C(\mathbb{R})$ consisting of all finitely supported functions, i.e. there exists a bounded interval $[\alpha, \beta] \subset \mathbb{R}$, such that $f(x) = 0$, $x \notin [\alpha, \beta]$
$\mathrm{supp}^c f = [\alpha, \beta]$	the convex-hull of the support of f, i.e. if $f \in C_0(\mathbb{R})$ is such that $f(x) = 0$ for $x \leq \alpha$ or $x \geq \beta$, with also $\alpha := \inf\{x : f(x) \neq 0\}$, $\beta := \sup\{x : f(x) \neq 0\}$
$C^k(\mathbb{R})$	for $k = 0, 1, \ldots$, $C^k(\mathbb{R}) := \{f \in M(\mathbb{R}) : f^{(j)} \in C(\mathbb{R}),\ j = 0, \ldots, k\}$, with the convention $f^{(0)} = f$, and where $C^0(\mathbb{R}) = C(\mathbb{R})$
$C_u(\mathbb{R})$	the linear space of uniformly bounded functions in $C(\mathbb{R})$
$\lVert \cdot \rVert_\infty$	the sup norm for the linear space $\ell^\infty(\mathbb{Z})$ (or $C_u(\mathbb{R})$), that is, $\lVert \mathbf{c} \rVert_\infty := \sup_j \lvert \mathbf{c}_j \rvert$, $\mathbf{c} \in \ell^\infty(\mathbb{Z})$, (or $\lVert f \rVert_\infty := \sup_x \lvert f(x) \rvert$, $f \in C_u(\mathbb{R})$)
$\boldsymbol{\Phi}_\mathbf{c}$	limit function of the subdivision scheme with subdivision operator $S_\mathbf{a}$ associated with the control point sequence $\{\mathbf{c}_j\} \in M^s(\mathbb{Z})$
ϕ	refinable function with respect to a given mask
π_k	the linear space of polynomials of degree $\leq k$
δ_j	the Kronecker delta, equal to zero for all $j \in \mathbb{Z}$, except for $\delta_0 = 1$
$\boldsymbol{\delta}$	the sequence $\{\delta_j : j \in \mathbb{Z}\}$
$\mathcal{A}_{m,n}$	the class of symmetric interpolatory mask symbol Laurent polynomials A, with A possessing a zero of order m at -1, and where $a_j = 0$, $j \notin \{-2n+1, \ldots, 2n-1\}$, $a_{-2n+1} \neq 0$, $a_{2n-1} \neq 0$
$T_\mathbf{a}$	cascade operator corresponding to a given mask $\mathbf{a} \in M_0(\mathbb{Z})$
$H^\alpha(\mathbb{R})$	for $\alpha \in (0, 1]$, the Lipschitz space $H^\alpha(\mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{R} : \lvert f(x) - f(y) \rvert \leq c \lvert x - y \rvert^\alpha,\ x, y \in \mathbb{R}$, for some constant $c \in [0, \infty)\}$
$H_0^\alpha(\mathbb{R})$	$H^\alpha(\mathbb{R}) \cap C_0(\mathbb{R})$, is a subspace of the space $C_0(\mathbb{R})$ of compactly supported continuous functions. For $\alpha = 1$, we also call $\mathrm{Lip}(\mathbb{R}) := H^1(\mathbb{R})$ the class of all Lipschitz continuous functions on $\mathbb{R}$
$C^{k,\alpha}(\mathbb{R})$	for $k = 0, 1, \ldots$, and $\alpha \in (0, 1]$, the Hölder space of order k with Hölder continuity exponent α is defined by $C^{k,\alpha}(\mathbb{R}) := \left\{f \in C^k(\mathbb{R}) : f^{(k)} \in H^\alpha(\mathbb{R})\right\}$
$C_0^{k,\alpha}(\mathbb{R})$	$C^{k,\alpha}(\mathbb{R}) \cap C_0(\mathbb{R})$, and observe that $C^{0,\alpha}(\mathbb{R}) = H^\alpha(\mathbb{R})$ and $C_0^{0,\alpha}(\mathbb{R}) = H_0^\alpha(\mathbb{R})$

1 Interpolatory Subdivision and Refinable Functions

In this chapter, after presenting a brief overview of the thesis and introducing a general class of symmetric interpolatory subdivision schemes that satisfies a polynomial reproduction property, we define a corresponding general class of symmetric interpolatory mask symbols. Also, we introduce the concept of an interpolatory refinable function, and develop some of its properties.

1.1 A brief overview and preliminaries

1.1.1 Overview

Subdivision is an important iterative technique for the efficient generation of curves and surfaces in geometric modelling. Both interpolatory and corner cutting algorithms find many applications in industry, for example in animation and graphics [9, 10, 31], cloth modelling [2, 29] and various others [32, 21, 30]. In this thesis the convergence of a class of symmetric interpolatory subdivision schemes for curves is studied.

Whereas convergence of a subdivision scheme implies the existence of a corresponding refinable function, the converse is not necessarily true [27]. In the case of interpolatory subdivision schemes, however, the converse is indeed true, i.e., an interpolatory subdivision scheme is convergent if and only if its associated refinable function exists [8]. The refinable function can, in turn, be used in the multi-resolutional construction method for wavelets and used in signal analysis applications, see for example [4, 23].

According to a result in [25], interpolatory subdivision schemes are guaranteed to converge if their mask symbol is positive on the unit circle of the complex plane . In this thesis we present a set of alternative conditions on symmetric interpolatory subdivision masks that is sufficient to guarantee the convergence of the associated subdivision scheme. These conditions are based on the contractivity of the subdivision operator corresponding to the residual Laurent polynomial after some integer power of the factor $1 + z$ has been factored out of the original Laurent polynomial mask symbol, but differ from the contractivity results of [14] (see also [15] and [17]).

First, in Sections 1.2 and 1.3, we develop a general class $\mathcal{A}_{m,n}$ of symmetric Laurent polynomial symbols A such that, for a given integer $m \in \mathbb{N}$, a polynomial reproduction property of degree $\leq 2m - 1$ is satisfied by the corresponding interpolatory subdivision scheme, and where the integer $n \geq m$ refers to the length of the coefficient sequence of A, which in turn is a measure of the locality of the corresponding subdivision scheme. Next, we introduce, in Section 1.4, the concept of an interpolatory refinable function ϕ, the existence of which is shown to imply the convergence of the corresponding interpolatory subdivision scheme, and we provide some of its properties.

In Chapters 2 and 3, we proceed to establish a useful sufficient condition on symbols $A \in \mathcal{A}_{m,n}$ such that there exists a corresponding interpolatory refinable function ϕ , and therefore the associated interpolatory subdivision scheme is convergent. First, in Section 2.1, we define the contractivity of a subdivision operator, and then use it in Section 2.2 as a convergence criterion in Theorem 2.5. We proceed in Section 2.3 to specialize the subdivision convergence result of Theorem 2.5 to interpolatory subdivision schemes with Laurent polynomial symbols $A \in \mathcal{A}_{m,n}$, as formulated in Theorem 2.6.

We proceed to develop, in Chapter 3, an interpolatory subdivision convergence theory, based on cascade algorithm convergence, with main result given by Theorem 3.7, which replaces, subject also to further restrictive condition, the contractivity condition of Theorem 2.6 with a weaker (contractivity) condition for interpolatory refinable function existence and corresponding interpolatory subdivision convergence.

In Chapter 4, after introducing a class of optimally local symmetric interpolatory subdivision schemes satisfying a certain polynomial reproduction property, which is often referred to as Dubuc-Deslauriers subdivision, as introduced and analyzed in [12, 11], we present, in Theorems 4.2 and 4.3, as proved in [25], a general interpolatory refinable function existence and corresponding subdivision convergence result for mask symbols $A \in \mathcal{A}_{m,n}$, with $A(z)$ strictly positive for z on the unit circle in complex plane. We proceed in Section 4.3 to give, in Theorem 4.6, an alternative proof of Theorem 4.3, by using our convergence result of Theorem 3.7, with the view also to make the interpolatory refinable function existence theory presented in this thesis self-contained. The result of Theorem 4.6 then enables us to remove the above mentioned restrictive condition in Theorem 3.7, and thereby yielding, in our main result of this thesis in Theorem 4.7, a substantial improvement on Theorem 3.7 for mask symbols $A \in \mathcal{A}_{m,n}$.

Finally, in Chapter 5, we apply the theory developed in Chapter 4 to analyze the convergence of a one-parameter family of interpolatory subdivision schemes which represents all of $\mathcal{A}_{m,m+1}$. In particular, we show that, for $m = 1, 2, 3$, the subdivision convergence result of our Theorem 4.7 yields a larger parameter interval for convergence than is the case when applying Theorem 4.2. In fact, for $m = 2, 3$, and especially for negative parameter values, an application of Theorem 4.7 improves, as far as we know, on all other analogous results in the literature. In addition, we present some Hölder regularity results for the corresponding interpolatory refinable function. Graphical illustrations for selected values of the parameter are provided by using Algorithms and initial control points in [5].

1.1.2 Preliminaries

Let $\{\mathbf{c}_j\}$, with $\mathbf{c}_j \in \mathbb{R}^s$, $j \in \mathbb{Z}$ for $s = 1, 2$ or 3, denote a given bi-infinite sequence. We shall call $\{\mathbf{c}_j\}$ a control point sequence.

An ***interpolatory subdivision scheme*** is an algorithm which generates, for $r = 0, 1, \ldots,$ the sequences $\{\mathbf{c}_j^{(r)}\}$ with $\mathbf{c}_j^{(r)} \in \mathbb{R}^s$, $j \in \mathbb{Z}$, recursively by means of

$$\mathbf{c}_j^{(0)} := \mathbf{c}_j \qquad ; \qquad \begin{cases} \mathbf{c}_{2j}^{(r+1)} := \mathbf{c}_j^{(r)}, \\ \\ \mathbf{c}_{2j+1}^{(r+1)} := \sum_{k=-n}^{n-1} w_k \mathbf{c}_{j-k}^{(r)}, \end{cases} \tag{1.1}$$

for all $j \in \mathbb{Z}$, and with $\{w_{-n}, \ldots, w_{n-1}\}$ denoting a set of $2n$ real-valued weights.

In this thesis, we consider only weight sequences satisfying the ***symmetry condition***

$$w_{-n+j} = w_{n-1-j}, \quad j = 0, \ldots, 2n-1, \tag{1.2}$$

as well as, for some $m \in \mathbb{N}$, the π_{2m-1}-***polynomial reproduction property***

$$\sum_{k=-n}^{n-1} w_k p(j-k) = p\left(j + \tfrac{1}{2}\right), \quad j \in \mathbb{Z}, \quad p \in \pi_{2m-1}, \tag{1.3}$$

or equivalently,

$$\sum_{k=-n}^{n-1} w_k (j-k)^\ell = \left(j + \tfrac{1}{2}\right)^\ell, \quad j \in \mathbb{Z}, \quad \ell = 0, \ldots, 2m-1, \tag{1.4}$$

where, for any non-negative integer k, π_k denotes the space of all polynomials of degree $\leq k$. Observe that we may choose $\ell = 0$ in (1.4) to deduce that

$$\sum_{k=-n}^{n-1} w_k = 1, \tag{1.5}$$

which is a natural condition on a sequence $\{w_{-n}, \ldots, w_{n-1}\}$ to qualify as a weight sequence.

A simple example is provided by choosing $n = 1$ and $\{w_{-1}, w_0\} = \left\{\frac{1}{2}, \frac{1}{2}\right\}$, in which case (1.1) is given by

$$c_j^{(0)} = c_j \qquad ; \qquad \begin{cases} c_{2j}^{(r+1)} = c_j^{(r)}, \\ \\ c_{2j+1}^{(r+1)} = \frac{1}{2} c_j^{(r)} + \frac{1}{2} c_{j+1}^{(r)}, \end{cases} \tag{1.6}$$

and, since, for any $j \in \mathbb{Z}$, we have here

$$w_{-1}(j+1) + w_0 j = \tfrac{1}{2}(j+1) + \tfrac{1}{2}j = j + \tfrac{1}{2},$$

the π_1-reproduction property is satisfied.

Observe that for any control point sequence $\{\mathbf{c}_j\}$ with $\mathbf{c}_j \in \mathbb{R}^s$ for $s = 2$, or 3, the interpolatory subdivision scheme (1.6) successively adds midpoints $\mathbf{c}_{2j+1}^{(r+1)}$ to the line segments connecting $\mathbf{c}_j^{(r)}$ and $\mathbf{c}_{j+1}^{(r)}$, and for $r \longrightarrow \infty$, therefore "fills up" the piecewise linear interpolant of the control points $\{\mathbf{c}_j\}$, as illustrated in Figure 1.1.

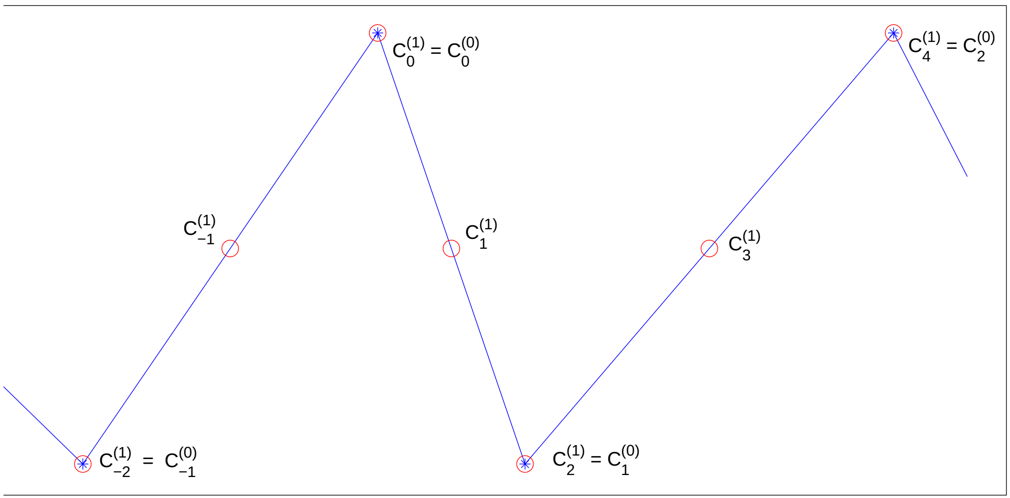


Figure 1.1: *Illustration of the interpolatory subdivision scheme (1.6), with initial sequence $\mathbf{c}^{(0)}$ denoted by $(*)$ and updated sequence $\mathbf{c}^{(1)}$ denoted by $(\circ)$.*

In the above example, it is clear that the sequence $\{\mathbf{c}_j^{(r)}\}$ approximately "doubles" in size for each successively increasing value of r, and also becomes denser at each step, eventually converging to the piecewise linear limit curve. In subdivision applications it is usually required that the limit curve be smooth, and in particular have no corners as in Figure 1.1. In this thesis, we study choices of $\{w_{-n}, \ldots, w_{n-1}\}$ for which such smoothness requirements are satisfied. In our subsequent analysis, we shall rely on the Laurent polynomial

$$A(z) := 1 + \sum_{j=-n}^{n-1} w_j z^{2j+1}, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{1.7}$$

called the **symbol** corresponding to the weights $\{w_{-n}, \dots, w_{n-1}\}$.

First, in Sections 1.2 and 1.3, we develop a general class $\mathcal{A}_{m,n}$ of symmetric Laurent polynomial symbols A such that, for a given integer $m \in \mathbb{N}$, the corresponding weights $\{w_{-n}, \dots, w_{n-1}\}$ satisfy the polynomial reproduction property (1.3), and where the integer $n \geq m$ refers to the length $\{-n, \dots, n-1\}$ of the weight sequence. Next,we introduce, in Section 1.4, the concept of an interpolatory refinable function ϕ, which is shown to be closely related to the convergence of the interpolatory subdivision scheme (1.1), and provide some of its properties.

1.2 Subdivision mask sequences

In this section, we proceed to characterize weight sequences $\{w_{-n}, \dots, w_{n-1}\}$ of the type introduced in Section 1.1. First, we introduce the following definition.

Definition 1.2.1

(a) *For* $s = 1, 2,$ *or* 3*, we denote by* $M^s(\mathbb{Z})$ *the space of all bi-infinite sequences* $\boldsymbol{c} = \{\boldsymbol{c}_j\} = \{\boldsymbol{c}_j : j \in \mathbb{Z}\}$ *defined on the set* $\mathbb{Z}$ *of all integers, with* $\boldsymbol{c}_j \in \mathbb{R}^s$*,* $j \in \mathbb{Z}$*, and we write* $M(\mathbb{Z})$ *for* $M^1(\mathbb{Z})$*.*

(b) *The subspace* $M_0^s(\mathbb{Z})$ *of* $M^s(\mathbb{Z})$ *is defined as those sequences with a finite number of non-zero elements, and we write* $M_0(\mathbb{Z})$ *for* $M_0^1(\mathbb{Z})$*.*

(c) *A sequence* $\mathbf{c} = \{c_j\} \in M_0(\mathbb{Z})$ *will be called a finitely supported sequence, in which case the support of* $\{c_j\}$ *is defined by*

$$\operatorname{supp}\{c_j\} := [\mu, \nu] \cap \mathbb{Z} = [\mu, \nu]|_{\mathbb{Z}}\,, \tag{1.8}$$

where

$$\mu := \min\{j : c_j \neq 0\} \qquad ; \qquad \nu := \max\{j : c_j \neq 0\}. \tag{1.9}$$

Our mathematical analysis of interpolatory subdivision schemes will be based on the following result.

THEOREM 1.1 *Let the sequence* $\{a_j\} \in M_0(\mathbb{Z})$ *satisfy the following conditions:*

(a)
$$a_{2j} = \delta_j, \qquad j \in \mathbb{Z}; \tag{1.10}$$

(b)
$$\mathrm{supp}\{a_j\} = [-2n+1, 2n-1]|_{\mathbb{Z}}, \qquad \textit{for an integer} \quad n \in \mathbb{N}; \tag{1.11}$$

(c)
$$a_{-j} = a_j, \qquad j \in \mathbb{Z}; \tag{1.12}$$

(d)
$$\sum_k a_{2j+1-2k} p(k) = p\left(j + \tfrac{1}{2}\right), \qquad j \in \mathbb{Z}, \quad p \in \pi_{2m-1}, \tag{1.13}$$

for an integer $m \in \mathbb{N}$.

Then, with
$$w_j := a_{2j+1}, \qquad j = -n, \dots, n-1, \tag{1.14}$$

the interpolatory subdivision scheme (1.1) has the equivalent formulation
$$c_j^{(0)} := c_j \qquad ; \qquad c_j^{(r+1)} := \sum_k a_{j-2k} c_k^{(r)}, \tag{1.15}$$

for $j \in \mathbb{Z}$ *and* $r = 0, 1, \dots,$ *and the weight sequence* $\{w_{-n}, \dots, w_{n-1}\}$ *satisfies the properties (1.2) and (1.3).*

Proof. For $r = 0, 1, \dots,$ and $j \in \mathbb{Z}$, it follows from (1.15) and (1.10) that
$$c_{2j}^{(r+1)} = \sum_k a_{2j-2k} c_k^{(r)} = \sum_k \delta_{j-k} c_k^{(r)} = c_j^{(r)}, \tag{1.16}$$

whereas, by using also (1.14) and (1.11),
$$\begin{aligned} c_{2j+1}^{(r+1)} = \sum_k a_{2j+1-2k} c_k^{(r)} &= \sum_k a_{2k+1} c_{j-k}^{(r)} \\ &= \sum_{k=-n}^{n-1} a_{2k+1} c_{j-k}^{(r)} \\ &= \sum_{k=-n}^{n-1} w_k c_{j-k}^{(r)}, \end{aligned}$$

so that (1.15) is indeed equivalent to (1.1).

Next, for $j \in \{0, \ldots, 2n-1\}$, we deduce from (1.14) and (1.12) that

$$\begin{aligned} w_{-n+j} = a_{2(-n+j)+1} &= a_{2(n-j)-1} \\ &= a_{2(n-1-j)+1} = w_{n-1-j}, \end{aligned}$$

which yields (1.2). Finally , we use (1.13), (1.11) and (1.14) to obtain, for $p \in \pi_{2m-1}$, and $j \in \mathbb{Z}$,

$$\begin{aligned} p\left(j + \tfrac{1}{2}\right) &= \sum_k a_{2k+1} p(j-k) \\ &= \sum_{k=-n}^{n-1} a_{2k+1} p(j-k) = \sum_{k=-n}^{n-1} w_k p(j-k), \end{aligned}$$

which proves (1.3). ■

Symmetric interpolatory subdivision schemes as in Theorem 1.1 were introduced in [11], [13] and [12].

On the basis of Theorem 1.1, we give the following definition.

Definition 1.2.2 *For any sequence* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$*, the operator* $S_{\boldsymbol{a}} : M^s(\mathbb{Z}) \longrightarrow M^s(\mathbb{Z})$ *is defined by*

$$\left(S_{\boldsymbol{a}} \boldsymbol{c}\right)_j := \sum_k a_{j-2k} \boldsymbol{c}_k, \qquad j \in \mathbb{Z}. \tag{1.17}$$

Observe from Definition 1.2.1 that the interpolatory subdivision scheme (1.1) has, with $\mathbf{a} = \{a_j\}$ and $\{w_{-n}, \ldots, w_{n-1}\}$ as in Theorem 1.1, the equivalent formulation,

$$\mathbf{c}_j^{(0)} := \mathbf{c}_j \qquad ; \qquad \mathbf{c}_j^{(r+1)} := \left(S_{\mathbf{a}} \mathbf{c}^{(r)}\right)_j = \left(S_{\mathbf{a}}^{r+1} \mathbf{c}\right)_j, \tag{1.18}$$

for $j \in \mathbb{Z}$ and $r = 0, 1, \ldots,$ where $\mathbf{c} = \{\mathbf{c}_j\}$, $\mathbf{c}^{(r)} = \{\mathbf{c}_j^{(r)}\}$, and $S_{\mathbf{a}}^0 \mathbf{c} := \mathbf{c}$.
The sequence $\{a_j\}$ will be called a ***subdivision mask sequence***, and the operator S_a will be referred to as the corresponding ***subdivision operator***. If, moreover, the sequence $\{a_j\}$ satisfies the condition (1.10), then $\{a_j\}$ will be called an ***interpolatory subdivision***

mask sequence, and $S_{\mathbf{a}}$ the corresponding ***interpolatory subdivision operator***.

Observe that the sequence $\{a_j\} \in M_0(\mathbb{Z})$ defined by

$$\{a_{-1}, a_0, a_1\} := \left\{\frac{1}{2}, 1, \frac{1}{2}\right\}, \qquad a_j = 0, \qquad |j| > 1, \tag{1.19}$$

satisfies the conditions $(a), \dots, (d)$ of Theorem 1.1 with $n = m = 1$, and that the definition (1.14) then yields precisely the interpolatory subdivision scheme (1.6).

1.3 Subdivision mask symbols

We proceed in this section to introduce a general class $\mathcal{A}_{m,n}$ of subdivision mask symbols that yield interpolatory subdivision schemes.

Definition 1.3.1 *Any function* $A : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ *of the form*

$$A(z) := \sum_{j=j_0}^{j_1} a_j z^j, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{1.20}$$

where $j_0 \in \mathbb{Z}$, *and* $j_1 \in \mathbb{Z}$, *is called a* ***Laurent polynomial****.*

Observe that if in Definition 1.3.1 the integer $j_0 \geq 0$, then A is a polynomial, in which case the origin 0 need not to be excluded from the definition (1.20) of A.
For a given subdivision mask sequence $\{a_j\} \in M_0(\mathbb{Z})$, the Laurent polynomial A defined by

$$A(z) := \sum_j a_j z^j, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{1.21}$$

will be called the ***subdivision mask symbol*** corresponding to $\{a_j\}$. Observe that, in the case of a mask sequence $\{a_j\}$ satisfying (1.10) and (1.11), the definition (1.14) shows that (1.21) and (1.7) yield precisely the same Laurent polynomial.

The conditions $(a), \ldots, (d)$ on $\{a_j\}$ in Theorem 1.1 have the following equivalent formulations, as given in [8] in terms of its corresponding symbol A, in which, for an integer $k \geq 0$, $A^{(k)}$ denotes the $k-$th derivative of the Laurent polynomial A, with $A^{(0)} := A$.

THEOREM 1.2 *A sequence* $\{a_j\} \in M_0(\mathbb{Z})$ *satisfies the conditions* $(a), \ldots, (d)$ *of Theorem 1.1 if and only if the corresponding Laurent polynomial symbol* A*, as given by (1.21), satisfies the following conditions:*

(a)

$$A(z) + A(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}; \tag{1.22}$$

(b)

$$A(z) = \sum_{j=-2n+1}^{2n-1} a_j z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{1.23}$$

with $a_{-2n+1} \neq 0$ *and* $a_{2n-1} \neq 0$*, for an integer* $n \in \mathbb{N}$*;*

(c)

$$A(z^{-1}) = A(z), \quad z \in \mathbb{C} \setminus \{0\}; \tag{1.24}$$

(d)

$$A(1) = 2, \tag{1.25}$$

$$A^{(k)}(-1) = 0, \quad k = 0, \ldots, 2m-1, \tag{1.26}$$

for an integer $m \in \mathbb{N}$*.*

Proof. (a) First use (1.21) to rewrite the left-hand side of (1.22), for $z \in \mathbb{C} \setminus \{0\}$, as

$$\begin{aligned} A(z) + A(-z) &= \sum_j a_j z^j + \sum_j a_j (-z)^j \\ &= \left[\sum_j a_{2j} z^{2j} + \sum_j a_{2j+1} z^{2j+1}\right] + \left[\sum_j a_{2j} z^{2j} - \sum_j a_{2j+1} z^{2j+1}\right], \end{aligned}$$

and thus

$$A(z) + A(-z) = 2 \sum_j a_{2j} z^{2j}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{1.27}$$

Now suppose that (1.10) holds. Then (1.27) gives

$$A(z) + A(-z) = 2\sum_j \delta_j z^{2j} = 2, \quad z \in \mathbb{C} \setminus \{0\},$$

so that (1.22) holds. If (1.22) holds, then (1.27) implies

$$2 = A(z) + A(-z) = 2\sum_j a_{2j} z^{2j}, \quad z \in \mathbb{C} \setminus \{0\},$$

and thus

$$\sum_j a_{2j} z^{2j} = 1, \quad z \in \mathbb{C} \setminus \{0\},$$

giving (1.10), and thereby establishing the fact that (1.10) and (1.22) are equivalent.

(b) Since (1.21) holds, we see that (1.11) and (1.23) are equivalent.

(c) By using (1.21), we obtain

$$A(z^{-1}) = \sum_j a_j z^{-j} = \sum_j a_{-j} z^j, \quad z \in \mathbb{C} \setminus \{0\},$$

which together with (1.21), shows that (1.12) and (1.24) are equivalent.

(d) First, observe from (1.21) that

$$A(z) = \sum_j a_{2j} z^{2j} + \sum_j a_{2j+1} z^{2j+1}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{1.28}$$

Suppose now the sequence $\{a_j\} \in M_0(\mathbb{Z})$ satisfies the conditions (a) and (d) of Theorem 1.1. It follows from (1.10)and (1.28) that

$$A(z) = 1 + \sum_j a_{2j+1} z^{2j+1}, \quad z \in \mathbb{C} \setminus \{0\}, \tag{1.29}$$

and thus

$$A(1) = 1 + \sum_j a_{2j+1}. \tag{1.30}$$

By choosing $p(x) = 1$, $x \in \mathbb{R}$, in (1.13), we obtain, for any $j \in \mathbb{Z}$,

$$1 = \sum_j a_{2j+1-2k} = \sum_k a_{2k+1}, \tag{1.31}$$

and it follows from (1.30) and (1.31) that (1.25) holds.

Also, (1.29) and (1.31) yield

$$A(-1) = 1 - \sum_j a_{2j+1} = 1 - 1 = 0,$$

which proves (1.26) for $k = 0$.

Now let $k \in \{1, \dots, 2m-1\}$. By differentiating (1.29) k times, we obtain the formula

$$A^{(k)}(z) = \sum_j p_k(j) a_{2j+1} z^{2j+1-k}, \quad z \in \mathbb{C} \setminus \{0\}, \tag{1.32}$$

where

$$p_k(x) = \prod_{\ell=-1}^{k-2} (2x - \ell), \quad x \in \mathbb{R}, \tag{1.33}$$

according to which $p_k \in \pi_k \subset \pi_{2m-1}$ and

$$p_k\Big(-\frac{1}{2}\Big) = 0. \tag{1.34}$$

It follows from (1.32) that

$$\begin{aligned} A^{(k)}(-1) &= (-1)^{k+1} \sum_j a_{2j+1} p_k(j) \\ &= (-1)^{k+1} \sum_j a_{1-2j} p_k(-j) \\ &= (-1)^{k+1} \sum_j a_{1-2j} q_k(j), \end{aligned} \tag{1.35}$$

where

$$q_k(x) := p_k(-x), \quad x \in \mathbb{R},$$

so that $q_k \in \pi_{2m-1}$, and, from (1.34) and (1.35)

$$q_k\Big(\frac{1}{2}\Big) = 0. \tag{1.36}$$

Now we use (1.35), together with (1.13) with $p = q_k$ and $j = 0$, to deduce that

$$A^{(k)}(-1) = (-1)^{k+1} q_k\Big(\tfrac{1}{2}\Big) = 0,$$

which completes the proof of the fact that $(a), \ldots, (d)$ in Theorem 1.1 imply $(a), \ldots, (d)$ in Theorem 1.2.

Suppose next that the Laurent polynomial A satisfies (1.22), (1.25) and (1.26). It remains to prove that the sequence $\{a_j\}$ then satisfies the condition (1.13), or equivalently, the condition

$$\sum_k a_{2j+1-2k} k^\ell = \left(j + \tfrac{1}{2}\right)^\ell, \quad j \in \mathbb{Z}, \quad \ell = 0, \ldots, 2m-1. \tag{1.37}$$

To this end, we first observe from (1.28), (1.25), and (1.26) with $k = 0$, that

$$\begin{cases} 2 \;=\; A(1) = \sum_j a_j = \sum_j a_{2j} + \sum_j a_{2j+1}; \\ 0 \;=\; A(-1) = \sum_j a_j (-1)^j = \sum_j a_{2j} - \sum_j a_{2j+1}; \end{cases} \tag{1.38}$$

which is satisfied if and only if

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1. \tag{1.39}$$

Since, for any $j \in \mathbb{Z}$, (1.39) yields

$$\sum_j a_{2j+1-2k} = \sum_k a_{2k+1} = 1;$$

it follows that (1.37) is indeed satisfied for $\ell = 0$.

Next, let $\ell \in \{1, \ldots, 2m-1\}$. From the equivalence of (1.22) and (1.10), it follows from (1.28) that (1.29), and therefore also (1.35), are satisfied, and thus, from (1.26) and (1.33),

$$\sum_k a_{1-2k} q_\ell(k) = 0, \tag{1.40}$$

where

$$q_\ell(x) := \prod_{i=-1}^{\ell-2} (2x + i), \quad x \in \mathbb{R}. \tag{1.41}$$

The polynomial sequence $\{1, q_1, \ldots, q_\ell\}$ is a basis for π_ℓ, and thus there exists a unique coefficient sequence $\{c_0, \ldots, c_\ell\} \subset \mathbb{R}$ such that

$$x^\ell = c_0 + \sum_{j=1}^{\ell} c_j q_j(x), \quad x \in \mathbb{R}, \tag{1.42}$$

in which we now set $x = \frac{1}{2}$ to deduce from (1.36) that $c_0 = \left(\frac{1}{2}\right)^\ell$, and thus

$$x^\ell = \left(\frac{1}{2}\right)^\ell + \sum_{j=1}^{\ell} c_j q_j(x), \quad x \in \mathbb{R}. \tag{1.43}$$

It follows from (1.43) and (1.40) that

$$\begin{aligned}
\sum_k a_{1-2k} k^\ell &= \sum_k a_{1-2k}\Big[\left(\frac{1}{2}\right)^\ell + \sum_{j=1}^{\ell} c_j q_j(k)\Big] \\
&= \left(\frac{1}{2}\right)^\ell \sum_k a_{1-2k} + \sum_{j=1}^{\ell} c_j \Big[\sum_k a_{1-2k} q_j(k)\Big] \\
&= \left(\frac{1}{2}\right)^\ell + \sum_{j=1}^{\ell} c_j(0) = \left(\frac{1}{2}\right)^\ell,
\end{aligned} \tag{1.44}$$

since also $\sum_k a_{1-2k} = \sum_k a_{2k+1} = 1$, from (1.39). Now let $j \in \mathbb{Z}$. It follows from (1.44) that

$$\begin{aligned}
\sum_k a_{2j+1-2k} k^\ell &= \sum_k a_{1-2k} (j+k)^\ell \\
&= \sum_k a_{1-2k} \sum_{i=0}^{\ell} \binom{\ell}{i} j^{\ell-i} k^i \\
&= \sum_{i=0}^{\ell} \binom{\ell}{i} j^{\ell-i} \sum_k a_{1-2k} k^i \\
&= \sum_{i=0}^{\ell} \binom{\ell}{i} j^{\ell-i} \left(\frac{1}{2}\right)^i \\
&= \left(j + \frac{1}{2}\right)^\ell,
\end{aligned}$$

which proves that (1.37) is also satisfied for $\ell = 1, \ldots, 2m-1$, and thereby completing our proof of the fact that conditions $(a), \ldots, (d)$ of Theorem 1.2 imply conditions $(a), \ldots, (d)$ of Theorem 1.1

■

Observe from (1.26) that a Laurent polynomial A as in Theorem 1.2 must contain the factor $(1+z)^{2m}$.

Motivated by Theorem 1.2, and following the definition introduced in [8], (see also [19, Definition 2.4], [18, Definition 2.9], as well as [26]), we now give the following definition.

DEFINITION 1.3.2 *For any positive integers* m *and* n*, the symbol* $\mathcal{A}_{m,n}$ *denotes the class of all Laurent polynomials* A *satisfying the conditions* $(a), \dots, (d)$ *of Theorem 1.2.*

As an example, consider the sequence $\{a_j\} \in M_0(\mathbb{Z})$ defined by (1.19), for which, from (1.21), the corresponding Laurent polynomial symbol is given by

$$A(z) = A_1(z) = \tfrac{1}{2}z^{-1} + 1 + \tfrac{1}{2}z = \tfrac{1}{2}z^{-1}(1+z)^2, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{1.45}$$

and thus, according to the Definition 1.3.2,

$$A_1 \in \mathcal{A}_{1,1}. \tag{1.46}$$

In order to investigate, in Chapter 3 below, the convergence of subdivision schemes with mask symbols in $\mathcal{A}_{m,n}$, we proceed to introduce, in the next Section 1.4, the concept of an interpolatory refinable function.

1.4 Interpolatory refinable functions and subdivision convergence

In this section, we introduce the concept of an interpolatory refinable function ϕ, and prove its close relationship with the convergence of its corresponding interpolatory subdivision scheme.

DEFINITION 1.4.1

(a) *The symbol* $C(\mathbb{R})$ *will denote the space of all real-valued continuous functions on* $\mathbb{R}$*.*

(b) *The subspace* $C_0(\mathbb{R})$ *of* $C(\mathbb{R})$ *is defined as those functions vanishing identically outside of some bounded closed (or compact) intervals of* $\mathbb{R}$*.*

(c) *A function* $f \in C_0(\mathbb{R})$ *will be called a compactly supported continuous function on* $\mathbb{R}$*, in which case the closure of the convex hull of the support of* f *will be denoted by*

$$\text{supp}^c f = [\alpha, \beta], \tag{1.47}$$

if $f(x) = 0$ *for* $x \leq \alpha$ *or* $x \geq \beta$, *with also*

$$\alpha := \inf\{x : f(x) \neq 0\} \quad ; \quad \beta := \sup\{x : f(x) \neq 0\}. \tag{1.48}$$

The convergence of the interpolatory subdivision scheme (1.15) is closely related to the concept of an interpolatory refinable function, which is defined as follows.

DEFINITION 1.4.2 *Let the sequence* $\{a_j\} \in M_0(\mathbb{Z})$ *and the function* $\phi \in C_0(\mathbb{R})$ *satisfy the equation*

$$\phi(x) = \sum_j a_j \phi(2x - j), \quad x \in \mathbb{R}. \tag{1.49}$$

Then ϕ *is called a* ***refinable function*** *with* ***refinement sequence*** $\{a_j\}$, *and (1.49) is called the corresponding* ***refinement equation***. *If, moreover,* ϕ *satisfies the condition*

$$\phi(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{1.50}$$

then ϕ *is called an* ***interpolatory refinable function***.

An example is provided by the hat function $\phi = h$, as defined by

$$h(x) = \begin{cases} 1 + x & , \quad -1 < x \leq 0, \\ \\ 1 - x & , \quad 0 \leq x < 1, \\ \\ 0 & , \quad x \in \mathbb{R} \setminus (-1, 1), \end{cases} \tag{1.51}$$

which satisfies the identity

$$h(x) = \tfrac{1}{2}h(2x + 1) + h(2x) + \tfrac{1}{2}h(2x - 1), \quad x \in \mathbb{R}, \tag{1.52}$$

and thus, since also $h(j) = \delta_j, \quad j \in \mathbb{Z}$, we see that $\phi = h$ is an interpolatory refinable function with refinement sequence $\{a_j\}$ given by (1.19). Observe that $h \in C_0(\mathbb{R}) \setminus C^1(\mathbb{R})$. The graph of h is shown in Figure 1.2

The following result can now be proved.

THEOREM 1.3 *The refinement sequence* $\{a_j\}$ *of an interpolatory refinable function satisfies the condition (1.10).*

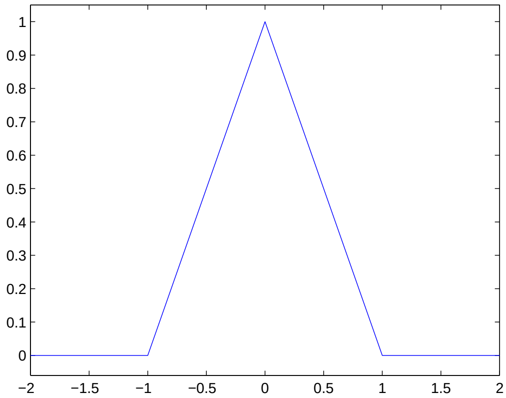


Figure 1.2: *The hat function h.*

Proof. By using (1.49) and (1.50), we obtain, for any $k \in \mathbb{Z}$,

$$\delta_k = \phi(k) = \sum_j a_j \phi(2k - j) = \sum_j a_{2k-j}\phi(j) = \sum_j a_{2k-j}\delta_j = a_{2k},$$

which completes our proof. ∎

The following uniqueness result shows that a sequence $\{a_j\} \in M_0(\mathbb{Z})$ satisfying (1.10) can have at most one corresponding interpolatory refinable function.

THEOREM 1.4 *Let* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ *be a sequence such that the condition (1.10) is satisfied, and suppose* $\phi \in C_0(\mathbb{R})$ *and* $\tilde{\phi} \in C_0(\mathbb{R})$ *are such that*

$$\left.\begin{aligned} \phi(x) &= \sum_j a_j \phi(2x - j), \quad x \in \mathbb{R}, & \widetilde{\phi}(x) &= \sum_j a_j \widetilde{\phi}(2x - j), \quad x \in \mathbb{R}, \\ \phi(j) &= \delta_j, \quad j \in \mathbb{Z}; & \widetilde{\phi}(j) &= \delta_j, \quad j \in \mathbb{Z}. \end{aligned}\right\} \tag{1.53}$$

Then

$$\phi = \widetilde{\phi}. \tag{1.54}$$

Proof. For the function $\phi^* \in C_0(\mathbb{R})$ defined by $\phi^* := \phi - \widetilde{\phi}$, it follows from the first line of (1.53) that, for any $x \in \mathbb{R}$,

$$\begin{aligned}\sum_j a_j \phi^*(2x-j) &= \sum_j a_j \phi(2x-j) - \sum_j a_j \widetilde{\phi}(2x-j) \\ &= \phi(x) - \widetilde{\phi}(x) = \phi^*(x). \end{aligned} \tag{1.55}$$

We claim that

$$\phi^*\left(\frac{j}{2^r}\right) = 0, \quad j \in \mathbb{Z}, \quad r = 0, 1, \dots, \tag{1.56}$$

which, since the dyadic sequence $\left\{\frac{j}{2^r} : j \in \mathbb{Z}, \quad r = 0, 1, \dots, \right\}$ is dense in $\mathbb{R}$, and since ϕ^* is a continuous function on $\mathbb{R}$, will imply that $\phi^*(x) = 0, x \in \mathbb{R}$, that is $\phi(x) - \widetilde{\phi}(x) = 0, x \in \mathbb{R}$, and thereby completing our proof.

Hence it remains to prove (1.56). To this end, we use (1.55) and (1.17), together with the assumption that the second line of (1.53) implies

$$\phi^*(j) = \phi(j) - \widetilde{\phi}(j) = 0, \quad j \in \mathbb{Z},$$

to deduce that, for any $j \in \mathbb{Z}$ and $r = 1, 2, \dots,$

$$\begin{aligned}\phi^*\left(\frac{j}{2^r}\right) &= \sum_k a_k \phi^*\left(\frac{j}{2^{r-1}} - k\right) \\ &= \sum_k a_k \sum_\ell a_\ell \phi^*\left(\frac{j}{2^{r-2}} - 2k - \ell\right) \\ &= \sum_k a_k \sum_\ell a_{\ell-2k} \phi^*\left(\frac{j}{2^{r-2}} - \ell\right) \\ &= \sum_\ell \left[\sum_k a_{\ell-2k} a_k\right] \phi^*\left(\frac{j}{2^{r-2}} - \ell\right) \\ &= \sum_\ell \left(S_a a\right)_\ell \phi^*\left(\frac{j}{2^{r-2}} - \ell\right) \\ &= \cdots \\ &= \sum_\ell \left(S_a^{r-1} a\right)_\ell \phi^*(j-\ell) \\ &= \sum_\ell \left(S_a^{r-1} a\right)_{j-\ell} \phi^*(\ell) \\ &= \sum_\ell \left(S_a^{r-1} a\right)_{j-\ell} (0) = 0, \end{aligned}$$

which proves the desired result (1.56). ■

Next, we establish the following relationship between the values at the half-integer of an interpolatory refinable function and its refinement sequence.

THEOREM 1.5 *Let ϕ denote an interpolatory refinable function with refinement sequence $\{a_j\}$. Then*

$$a_j = \phi\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}. \tag{1.57}$$

Proof. From (1.49) and (1.50), we have, for any $j \in \mathbb{Z}$,

$$\phi\left(\frac{j}{2}\right) = \sum_k a_k \phi(j-k) = \sum_k a_k \delta_{j-k} = a_j,$$

which completes our proof.

■

In general the existence of a refinable function does not guarantee the convergence of the associated subdivision scheme, see e.g. [3, 27, 20]. Here however, as proved in [8, Theorem 2], for the interpolatory subdivision scheme (1.18), the existence of an interpolatory refinable function ϕ with refinement sequence $\{a_j\}$ guarantees subdivision convergence, as follows.

THEOREM 1.6 *Let ϕ denote an interpolatory refinable function with refinement sequence $\{a_j\}$. Then the interpolatory subdivision scheme (1.18) satisfies, for any control point sequence $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in M^s(\mathbb{Z})$, with $s = 1, 2,$ or 3,*

$$\boldsymbol{c}_j^{(r)} = \boldsymbol{\Phi}_{\boldsymbol{c}}\left(\frac{j}{2^r}\right), \qquad j \in \mathbb{Z}, \quad r = 0, 1, \ldots, \tag{1.58}$$

where

$$\boldsymbol{\Phi}_{\boldsymbol{c}}(x) := \sum_j \boldsymbol{c}_j \phi(x-j), \qquad x \in \mathbb{R}. \tag{1.59}$$

*Also, $\boldsymbol{\Phi}_{\boldsymbol{c}} : \mathbb{R} \longrightarrow \mathbb{R}^s$ is a continuous (vector-valued if $s \geq 2$) function on $\mathbb{R}$, and $\boldsymbol{\Phi}_{\boldsymbol{c}}$ is the **limit curve** of the interpolatory subdivision scheme (1.18), in the sense that, for any*

$x \in \mathbb{R}$, *and a sequence* $\{j_r : r = 0, 1, \ldots\} \subset \mathbb{Z}$ *such that*

$$\left| x - \frac{j_r}{2^r} \right| \longrightarrow 0, \quad r \longrightarrow \infty, \tag{1.60}$$

the convergence result

$$\left| \mathbf{\Phi}_{\boldsymbol{c}}(x) - \boldsymbol{c}_{j_r}^{(r)} \right| \longrightarrow 0, \quad r \longrightarrow \infty, \tag{1.61}$$

holds.

Proof. For $j \in \mathbb{Z}$ and $r = 0, 1, \ldots,$ it follows from (1.59), (1.49), (1.18) and (1.50) that

$$\begin{aligned}
\mathbf{\Phi}_{\mathbf{c}}\Big(\frac{j}{2^r}\Big) &= \sum_k \mathbf{c}_k^{(0)} \phi\Big(\frac{j}{2^r} - k\Big) \\
&= \sum_k \mathbf{c}_k^{(0)} \sum_\ell a_\ell \phi\Big(\frac{j}{2^{r-1}} - 2k - \ell\Big) \\
&= \sum_k \mathbf{c}_k^{(0)} \sum_\ell a_{\ell-2k} \phi\Big(\frac{j}{2^{r-1}} - \ell\Big) \\
&= \sum_\ell \Big[\sum_k a_{\ell-2k} \mathbf{c}_k^{(0)}\Big] \phi\Big(\frac{j}{2^{r-1}} - \ell\Big) \\
&= \sum_\ell \mathbf{c}_\ell^{(1)} \phi\Big(\frac{j}{2^{r-1}} - \ell\Big) \\
&= \cdots \\
&= \sum_\ell \mathbf{c}_\ell^{(r)} \phi(j - \ell) \\
&= \sum_\ell \mathbf{c}_\ell^{(r)} \delta_{j-\ell} = \mathbf{c}_j^{(r)},
\end{aligned}$$

which proves (1.58).

Since ϕ is compactly supported, the summation in (1.59) is, for any fixed $x \in \mathbb{R}$, over a finite number of indexes j. Since ϕ is continuous for each $x \in \mathbb{R}$, it follows that $\mathbf{\Phi}_{\mathbf{c}}$ is continuous on $\mathbb{R}$.

Let $x \in \mathbb{R}$ be fixed. Since the dyadic sequence $\left\{\frac{j}{2^r} : j \in \mathbb{Z}, r = 0, 1, \ldots\right\}$ is dense in $\mathbb{R}$, there exists a sequence $\{j_r : r = 0, 1, \ldots\}$ such that (1.60) is satisfied. By using also (1.58), we obtain, for any $r = 0, 1, \ldots,$

$$\left| \mathbf{\Phi}_{\mathbf{c}}(x) - \mathbf{c}_{j_r}^{(r)} \right| = \left| \mathbf{\Phi}_{\mathbf{c}}(x) - \mathbf{\Phi}_{\mathbf{c}}\Big(\frac{j_r}{2^r}\Big) \right| \longrightarrow 0, \qquad r \longrightarrow \infty,$$

from (1.60) and the continuity of $\mathbf{\Phi_c}$ at x, and thereby completing our proof. ■

An example is provided by the choice $\phi = h$, the hat function given by (1.51), and which is an interpolatory refinable function with refinement sequence given by (1.19). According to Theorem 1.6, the limit curve $\mathbf{\Phi_c}$ of the corresponding interpolatory subdivision scheme (1.6) is given by

$$\mathbf{\Phi_c}(x) = \sum_j \mathbf{c}_j h(x - j), \quad x \in \mathbb{R}. \tag{1.62}$$

Since $h(j) = \delta_j$, $j \in \mathbb{Z}$, we see from (1.62) that, for any $k \in \mathbb{Z}$,

$$\mathbf{\Phi_c}(k) = \sum_j \mathbf{c}_j h(k - j) = \sum_j \mathbf{c}_j \delta_{k-j} = \mathbf{c}_k, \tag{1.63}$$

that is, if the control point sequence $\{\mathbf{c}_j\} \in M^s(\mathbb{Z})$ for $s = 2$, or 3, then $\mathbf{\Phi_c}$ is the linear interpolant of the sequence $\{\mathbf{c}_j\}$, and according to (1.58), (1.60), (1.61), the sequences $\{c_j^{(r)}\}$, $r = 0, 1, \ldots$, "fill up" the limit curve $\mathbf{\Phi_c}$, as noted before in Section 1.1 (see also Figure 1.1).

An interpolatory refinable function ϕ, with refinement sequence $\{a_j\}$ for which the corresponding Laurent polynomial A belongs to $\mathcal{A}_{m,n}$, possesses the following properties.

THEOREM 1.7 *Let ϕ denote an interpolatory refinable function with refinement sequence $\{a_j\}$ such that the corresponding Laurent polynomial A defined by (1.21) satisfies $A \in \mathcal{A}_{m,n}$. Then*

(a)

$$\text{supp}^c \phi = [-2n + 1, 2n - 1]; \tag{1.64}$$

(b)

$$\phi(-x) = \phi(x), \quad x \in \mathbb{R}; \tag{1.65}$$

(c)

$$\sum_j p(j)\phi(x - j) = p(x), \qquad x \in \mathbb{R}, \qquad p \in \pi_{2m-1}. \tag{1.66}$$

Proof. (a) The definitions

$$\mu := \inf\Big\{x : \phi(x) \neq 0\Big\} \quad ; \quad \nu := \sup\Big\{x : \phi(x) \neq 0\Big\}, \tag{1.67}$$

yield

$$\inf\Big\{x : \phi(2x+2n-1) \neq 0\Big\} = \frac{\mu - 2n + 1}{2}; \quad \sup\Big\{x : \phi(2x-2n+1) \neq 0\Big\} = \frac{\nu + 2n - 1}{2}. \tag{1.68}$$

Since property (b) in Theorem 1.2 gives $a_{-2n+1} \neq 0$ and $a_{2n-1} \neq 0$, it follows from (1.68) that

$$\inf\Big\{x : \sum_{j=-2n+1}^{2n-1} a_j \phi(2x-j) \neq 0\Big\} = \inf\Big\{x : \phi(2x+2n-1) \neq 0\Big\} = \frac{\mu - 2n + 1}{2},$$

$$\sup\Big\{x : \sum_{j=-2n+1}^{2n-1} a_j \phi(2x-j) \neq 0\Big\} = \sup\Big\{x : \phi(2x-2n+1) \neq 0\Big\} = \frac{\nu + 2n - 1}{2},$$

and thus, from (1.67) and the refinement equation (1.49), we have

$$\mu = \frac{\mu - 2n + 1}{2} \quad ; \quad \nu = \frac{\nu + 2n - 1}{2},$$

that is,

$$\mu = -2n + 1 \quad ; \quad \nu = 2n - 1,$$

which, together with (1.67), proves (1.64).

(b) Since ϕ is continuous on $\mathbb{R}$, and since the dyadic sequence $\Big\{\frac{j}{2^r} : j \in \mathbb{Z}, r = 0, 1, \ldots,\Big\}$ is dense in $\mathbb{R}$, it will suffice to prove that

$$\phi\Big(-\frac{j}{2^r}\Big) = \phi\Big(\frac{j}{2^r}\Big), \quad j \in \mathbb{Z}, \quad r = 0, 1, \ldots. \tag{1.69}$$

To this end, we first note from (1.50) that (1.69) holds for $r = 0$. Next, from the equivalence of (1.24) and (1.12), according to which $a_{-j} = a_j, j \in \mathbb{Z}$, we use (1.57) to deduce that (1.69) also holds for $r = 1$. For $r \geq 2$, we apply the refinement equation

(1.49), and $a_{-j} = a_j$, $j \in \mathbb{Z}$, to obtain, by using also (1.17), for any $j \in \mathbb{Z}$,

$$\begin{aligned}
\phi\Big(-\frac{j}{2^r}\Big) = \sum_k a_k \phi\Big(-\frac{j}{2^{r-1}} - k\Big) &= \sum_k a_{-k}\phi\Big(-\frac{j}{2^{r-1}} - k\Big) \\
&= \sum_k a_k \phi\Big(-\frac{j}{2^{r-1}} + k\Big) \\
&= \sum_k a_k \sum_\ell a_\ell \phi\Big(-\frac{j}{2^{r-2}} + 2k - \ell\Big) \\
&= \sum_k a_k \sum_\ell a_{2k-\ell} \phi\Big(-\frac{j}{2^{r-2}} + \ell\Big) \\
&= \sum_k a_k \sum_\ell a_{\ell-2k} \phi\Big(-\frac{j}{2^{r-2}} + \ell\Big) \\
&= \sum_\ell \Big[\sum_k a_{\ell-2k} a_k\Big] \phi\Big(-\frac{j}{2^{r-2}} + \ell\Big) \\
&= \sum_\ell \Big(S_a a\Big)_\ell \phi\Big(-\frac{j}{2^{r-2}} + \ell\Big) \\
&= \cdots \\
&= \sum_\ell \Big(S_a^{r-1} a\Big)_\ell \phi(-j + \ell) \\
&= \sum_\ell \Big(S_a^{r-1} a\Big)_\ell \delta_{-j+\ell} = \Big(S_a^{r-1} a\Big)_j.
\end{aligned} \tag{1.70}$$

Similarly, for any $j \in \mathbb{Z}$,

$$\begin{aligned}
\phi\Big(\frac{j}{2^r}\Big) &= \sum_k a_k \phi\Big(\frac{j}{2^{r-1}} - k\Big) \\
&= \sum_k a_k \sum_\ell a_\ell \phi\Big(\frac{j}{2^{r-2}} - 2k - \ell\Big) \\
&= \sum_k a_k \sum_\ell a_{\ell-2k} \phi\Big(\frac{j}{2^{r-2}} - \ell\Big) \\
&= \sum_\ell \Big[\sum_k a_{\ell-2k} a_k\Big] \phi\Big(\frac{j}{2^{r-2}} - \ell\Big) \\
&= \sum_\ell \Big(S_a a\Big)_\ell \phi\Big(\frac{j}{2^{r-2}} - \ell\Big) \\
&= \cdots \\
&= \sum_\ell \Big(S_a^{r-1} a\Big)_\ell \phi(j - \ell) \\
&= \sum_\ell \Big(S_a^{r-1} a\Big)_\ell \delta_{j-\ell} = \Big(S_a^{r-1} a\Big)_j.
\end{aligned} \tag{1.71}$$

It follows from(1.70) and (1.71) that (1.69) is indeed satisfied for all $j \in \mathbb{Z}$ and $r = 0, 1, \ldots,$ and thereby completing our proof of (b).

(c) Let $p \in \pi_{2m-1}$. Since $\sum_j p(j)\phi(\cdot - j)$ and p are continuous functions on $\mathbb{R}$, and since the dyadic sequence $\left\{\frac{k}{2^r} : k \in \mathbb{Z}, r = 0, 1, \ldots\right\}$ is dense in $\mathbb{R}$, it will suffice to prove that, for any $k \in \mathbb{Z}$, and $r = 0, 1, \ldots,$

$$\sum_j p(j)\phi\Big(\frac{k}{2^r} - j\Big) = p\Big(\frac{k}{2^r}\Big), \tag{1.72}$$

or equivalently,

$$\sum_j j^\ell \phi\Big(\frac{k}{2^r} - j\Big) = \Big(\frac{k}{2^r}\Big)^\ell, \quad \ell = 0, \ldots, 2m-1. \tag{1.73}$$

Since $A \in \mathcal{A}_{m,n}$, it follows from Theorem 1.2 that $\{a_j\}$ satisfies the condition (1.13), that is,

$$\sum_k k^\ell a_{2j+1-2k} = \Big(\frac{2j+1}{2}\Big)^\ell, \quad j \in \mathbb{Z}, \quad \ell = 0, \ldots, 2m-1. \tag{1.74}$$

Also, again from Theorem 1.2, the sequence $\{a_j\}$ satisfies (1.10), and thus

$$\sum_k k^\ell a_{2j-2k} = \sum_k k^\ell \delta_{j-k} = j^\ell = \Big(\frac{2j}{2}\Big)^\ell, \quad j \in \mathbb{Z}, \quad \ell = 0, \ldots, 2m-1. \tag{1.75}$$

By combining (1.74) and (1.75), we deduce that

$$\sum_k k^\ell a_{j-2k} = \Big(\frac{j}{2}\Big)^\ell, \quad j \in \mathbb{Z}, \quad \ell = 0, \ldots, 2m-1. \tag{1.76}$$

Let $k \in \mathbb{Z}$ and $r \in \{0, 1, \ldots\}$. By using (1.49), (1.76) and (1.50), we obtain

$$\begin{aligned}
\sum_j j^\ell \phi\Big(\frac{k}{2^r} - j\Big) &= \sum_j j^\ell \sum_i a_i \phi\Big(\frac{k}{2^{r-1}} - 2j - i\Big) \\
&= \sum_j j^\ell \sum_i a_{i-2j} \phi\Big(\frac{k}{2^{r-1}} - i\Big) \\
&= \sum_i \Big[\sum_j j^\ell a_{i-2j}\Big] \phi\Big(\frac{k}{2^{r-1}} - i\Big) \\
&= \sum_i \Big(\frac{i}{2}\Big)^\ell \phi\Big(\frac{k}{2^{r-1}} - i\Big) \\
&= \frac{1}{2^\ell} \sum_i i^\ell \phi\Big(\frac{k}{2^{r-1}} - i\Big) \\
&= \cdots \\
&= \frac{1}{(2^\ell)^r} \sum_i i^\ell \phi(k - i) \\
&= \frac{1}{2^{\ell r}} \sum_i i^\ell \delta_{k-i} = \frac{1}{2^{\ell r}} k^\ell = \Big(\frac{k}{2^r}\Big)^\ell,
\end{aligned}$$

which completes the proof of (1.73). ■

The following result is obtained immediately from Theorem 1.7(c) by choosing $p(x) = 1$, $x \in \mathbb{R}$, in (1.66).

Corollary 1.8 *Let ϕ be an interpolatory refinable function with refinement sequence as in Theorem 1.7. Then*

$$\sum_j \phi(x - j) = 1, \quad x \in \mathbb{R}. \tag{1.77}$$

The following result, for the proof of which we refer to [5, Theorem 2.1.1], shows that the support interval of a refinable function ϕ agrees with the support of its refinement sequence.

THEOREM 1.9 *Let* ϕ *be a refinable function with refinement sequence* $\{a_j\}$ *satisfying*

$$\mathrm{supp}\{a_j\} = [\mu, \nu]|_{\mathbb{Z}}\,. \tag{1.78}$$

Then

$$\mathrm{supp}^c \phi = [\mu, \nu]. \tag{1.79}$$

As a consequence, we immediately have the following.

COROLLARY 1.10 *Let* ϕ *be an interpolatory refinable function with refinement sequence as in Theorem 1.7. Then*

$$\mathrm{supp}^c \phi = [-2n + 1, 2n - 1]. \tag{1.80}$$

Recall the case where $\phi = h$, the hat function given in (1.51), which is an interpolatory refinable function with refinement sequence $\{a_j\}$ given by (1.19), according to which $\mathrm{supp}\{a_j\} = [-1, 1]|_{\mathbb{Z}}$, and thus, from Corollary 1.10, $\mathrm{supp}^c h = [-1, 1]$, as is also clear from (1.51).

Note that Corollary 1.10 is in accordance with Theorem 1.7 (a).

In Chapters 2 and 3, we proceed to establish a useful sufficient condition on sequences $\{a_j\} \in M_0(\mathbb{Z})$, with the corresponding symbol $A \in \mathcal{A}_{m,n}$, such that there exists a corresponding interpolatory refinable function ϕ , and therefore, according to Theorem 1.6, such that the associated interpolatory subdivision scheme is convergent. In particular, we shall obtain interpolatory refinable functions ϕ with better smoothness properties than the hat function h, resulting therefore, according to (1.59), in limit curves $\Phi_{\mathbf{c}}$ with corresponding better smoothness properties than the piecewise linear limit curve (1.62) of the interpolatory subdivision scheme (1.6).

Contractive Subdivision operators and Subdivision Convergence

2

In this chapter, after defining the contractivity of the subdivision operator and providing a general definition of subdivision convergence, we proceed to derive a sufficient condition, based on the contractivity of the subdivision operator, that assures the convergence of the subdivision scheme. We then apply this subdivision convergence result to the class $\mathcal{A}_{m,n}$.

2.1 Contractive subdivision operators

The following definitions of boundedness and contractivity are standard for operators on normed linear spaces (see e.g. in [3, 22]):

Definition 2.1.1 *Let* $(\mathcal{B}, \|\cdot\|)$ *be a normed linear space and suppose* $\mathtt{L}$ *is a linear operator mapping* $\mathcal{B}$ *into itself. We say that* $\mathtt{L}$ *is* ***bounded*** *if*

$$\|\mathtt{L}\| := \sup\left\{\frac{\|\mathtt{L}f\|}{\|f\|} : f \in \mathcal{B}, \quad f \neq 0\right\} < \infty. \tag{2.1}$$

If it moreover holds that

$$\|\mathtt{L}\| < 1, \tag{2.2}$$

we say that $\mathtt{L}$ *is* ***contractive****.*

Observe in particular from (2.1) that

$$\|\mathtt{L}f\| \leq \|\mathtt{L}\| \, \|f\|, \qquad f \in \mathcal{B}, \tag{2.3}$$

for any linear bounded operator $\mathtt{L}$ mapping a normed linear space $\mathcal{B}$ into itself.

DEFINITION 2.1.2 *The subspace* $\ell^\infty(\mathbb{Z})$ *of* $M^s(\mathbb{Z})$, *for* $s = 1, 2,$ *or* 3, *is defined as the class of all uniformly bounded sequences, that is,*

$$\|\boldsymbol{c}\|_\infty = \|\{\boldsymbol{c}_j\}\|_\infty := \sup_j |\boldsymbol{c}_j| < \infty,$$

with $|\boldsymbol{c}_j|$ *denoting the Euclidean length of the vector* $\boldsymbol{c}_j \in \mathbb{R}^s$.

Observe in particular that

$$M_0^s(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}).$$

Our next result provides an explicit formulation, as appearing e.g. in [3, 14, 17], for the $\infty-$norm of a subdivision operator.

THEOREM 2.1 *For a sequence* $\boldsymbol{b} = \{b_j\} \in M_0(\mathbb{Z})$, *the corresponding subdivision operator* $S_{\boldsymbol{b}}$, *as given for any control point sequence* $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in M^s(\mathbb{Z})$ *by*

$$\left(S_{\boldsymbol{b}}\boldsymbol{c}\right)_j := \sum_k b_{j-2k}\boldsymbol{c}_k, \quad j \in \mathbb{Z}, \tag{2.4}$$

is a bounded linear operator from $\ell^\infty(\mathbb{Z})$ *into itself, with*

$$\|S_{\boldsymbol{b}}\|_\infty = \rho_{\boldsymbol{b}} := \max\left\{\sum_j |b_{2j}|, \sum_j |b_{2j+1}|\right\}. \tag{2.5}$$

Also,

$$\|S_{\boldsymbol{b}}c\|_\infty \leq \rho_{\boldsymbol{b}} \|\boldsymbol{c}\|_\infty, \quad \boldsymbol{c} = \{\boldsymbol{c}_j\} \in \ell^\infty(\mathbb{Z}). \tag{2.6}$$

Proof. Let $\mathbf{c} = \{\mathbf{c}_j\} \in \ell^\infty(\mathbb{Z})$. Then, from (2.4), and for any $j \in \mathbb{Z}$,

$$\begin{aligned}\left|\left(S_{\mathbf{b}}\mathbf{c}\right)_{2j}\right| &= \left|\sum_k b_{2j-2k}\mathbf{c}_k\right| = \left|\sum_k b_{2k}\mathbf{c}_{j-k}\right| \\ &\leq \|\mathbf{c}\|_\infty \sum_k |b_{2k}|,\end{aligned}$$

and similarly,

$$\begin{aligned}\left|\left(S_{\mathbf{b}}\mathbf{c}\right)_{2j+1}\right| &= \left|\sum_k b_{2j+1-2k}\mathbf{c}_k\right| = \left|\sum_k b_{2k+1}\mathbf{c}_{j-k}\right| \\ &\leq \|\mathbf{c}\|_\infty \sum_k |b_{2k+1}|,\end{aligned}$$

and thus

$$\left|\left(S_{\mathbf{b}}\mathbf{c}\right)_j\right| \leq \rho_{\mathbf{b}} \left\|\mathbf{c}\right\|_\infty, \quad j \in \mathbb{Z}, \tag{2.7}$$

with $\rho_{\mathbf{b}}$ defined as in (2.5). It follows from (2.7) that (2.6) holds. Hence $S_{\mathbf{b}}$ maps $\ell^\infty(\mathbb{Z})$ into itself. Also, (2.4) shows that $S_{\mathbf{b}}$ is a linear operator on $M^s(\mathbb{Z}) \supset \ell^\infty(\mathbb{Z})$. Moreover, (2.6) implies that

$$\frac{\|S_{\mathbf{b}}\mathbf{c}\|_\infty}{\|\mathbf{c}\|_\infty} \leq \rho_{\mathbf{b}}, \tag{2.8}$$

for any non-zero sequence $\mathbf{c} = \{\mathbf{c}_j\} \in \ell^\infty(\mathbb{Z})$, and it follow from Definition 2.1.1 that $S_{\mathbf{b}}$ is bounded, with

$$\|S_{\mathbf{b}}\|_\infty \leq \rho_{\mathbf{b}}. \tag{2.9}$$

Finally, we will show that

$$\|S_{\mathbf{b}}\|_\infty \geq \rho_{\mathbf{b}}, \tag{2.10}$$

which, together with (2.9), then establishes (2.5). To this end, let $j \in \mathbb{Z}$ be fixed, and consider the sequence

$$\left.\begin{aligned} \mathbf{c}_j = \{\mathbf{c}_{j,k} : k \in \mathbb{Z}\} \;&:=\; \{(\tilde{c}_{j,k}, 0, 0, \ldots, 0) \in \mathbb{R}^s : k \in \mathbb{Z}\}, \\ \text{where} \qquad & \\ \tilde{c}_{j,k} \;&:=\; \begin{cases} 1 &, \text{ if } \; b_{j-2k} \geq 0, \\ -1 &, \text{ if } \; b_{j-2k} < 0. \end{cases} \end{aligned}\right\} \tag{2.11}$$

It follows from (2.11), (2.4) and (2.3) that, with $\mathbf{c}_j = \{\mathbf{c}_{j,k} : k \in \mathbb{Z}\}$,

$$\begin{aligned} \sum_k |b_{j-2k}| &= \sum_k b_{j-2k}\tilde{c}_{j,k} \\ &= \left|\sum_k b_{j-2k}\tilde{c}_{j,k}\right| \\ &= \left|\sum_k b_{j-2k}(\tilde{c}_{j,k}, 0, 0, \ldots, 0)\right| \\ &= \left|\left(S_{\mathbf{b}}\mathbf{c}_j\right)_j\right| \\ &\leq \sup_k \left|\left(S_{\mathbf{b}}\mathbf{c}_j\right)_k\right| \\ &= \|S_{\mathbf{b}}\mathbf{c}_j\|_\infty \leq \|S_{\mathbf{b}}\|_\infty \|\mathbf{c}_j\|_\infty = \|S_{\mathbf{b}}\|_\infty, \end{aligned} \tag{2.12}$$

since (2.11) yields $\|\mathbf{c}_j\|_\infty = 1$. But $\|S_{\mathbf{b}}\|_\infty$ is independent of j, so that (2.12) gives

$$\sup_j \sum_k |b_{j-2k}| \leq \|S_{\mathbf{b}}\|_\infty \, . \tag{2.13}$$

Now observe that

$$\begin{aligned} \sum_k |b_{2j-2k}| &= \sum_k |b_{2k}| \, , \quad j \in \mathbb{Z}, \\ \sum_k |b_{2j+1-2k}| &= \sum_k |b_{2k+1}| \, , \quad j \in \mathbb{Z}, \end{aligned}$$

according to which

$$\sup_j \sum_k |b_{j-2k}| = \max\left\{ \sum_k |b_{2k}| \, , \sum_k |b_{2k+1}| \right\}. \tag{2.14}$$

It then follows from (2.13), (2.14), together with the definition of $\rho_{\mathbf{b}}$ in (2.5), that the desired result (2.10) does indeed hold. ∎

2.2 Subdivision convergence results

A general definition for (not necessarily interpolatory) subdivision convergence is given by the following (see e.g. in [3, 14, 24]).

DEFINITION 2.2.1 *For a subdivision mask sequence* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ *and a control point sequence* $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in M^s(\mathbb{Z})$*, for* $s = 1, 2,$ *or* 3*, the subdivision scheme (1.18) is said to be convergent if there exists a non-trivial continuous function* $\boldsymbol{\Phi}_{\boldsymbol{c}} : \mathbb{R} \longrightarrow \mathbb{R}^s$ *such that*

$$\sup_j \left| \boldsymbol{\Phi}_{\boldsymbol{c}}\left(\frac{j}{2^r}\right) - \boldsymbol{c}_j^{(r)} \right| \longrightarrow 0, \qquad r \longrightarrow \infty. \tag{2.15}$$

Observe that the interpolatory subdivision convergence result of Theorem 1.6 satisfies Definition 2.2.1, in the sense that (1.58) trivially implies (2.15).

The following implications of subdivision convergence are proved in [3] and [14] (see also [5]).

THEOREM 2.2 *Suppose, for a subdivision mask sequence* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$*, the subdivision scheme (1.18), with control point sequence* $\boldsymbol{c} = \boldsymbol{\delta} = \{\delta_j\} \in M_0(\mathbb{Z})$*, is convergent. Then the following hold:*

(a)

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1; \tag{2.16}$$

(b) *the limit function* $\phi = \phi_{\boldsymbol{a}} := \Phi_{\boldsymbol{\delta}}$ *satisfies:*

(i) ϕ *is a refinable function with refinement sequence* $\{a_j\}$*;*

(ii)

$$\sum_j \phi(x-j) = 1, \qquad x \in \mathbb{R}; \tag{2.17}$$

(iii)

$$\int_{-\infty}^{\infty} \phi(x)dx = 1; \tag{2.18}$$

(c) *the subdivision scheme (1.18) converges for any control point sequence* $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in \ell^\infty(\mathbb{Z}) \subset M^s(\mathbb{Z})$*, and with limit function* $\boldsymbol{\Phi}_{\boldsymbol{c}}$ *given by*

$$\boldsymbol{\Phi}_{\boldsymbol{c}}(x) := \sum_j \boldsymbol{c}_j \phi(x-j), \qquad x \in \mathbb{R}. \tag{2.19}$$

As proved in the argument leading to (1.38), the sum rule (2.16) has the equivalent formulation

$$A(1) = 2 \qquad ; \qquad A(-1) = 0, \tag{2.20}$$

in terms of the corresponding Laurent polynomial symbol defined in (1.21). It follows that the sum rule condition (2.16) is satisfied if and only if

$$A(z) = (1+z)B(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{2.21}$$

for some Laurent polynomial B satisfying

$$B(1) = 1. \tag{2.22}$$

The following necessary and sufficient condition for subdivision convergence was proved in [15, Theorem 3] (see also [17, Theorem 4.11]).

THEOREM 2.3 *Let* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ *be a subdivision mask sequence such that its corresponding Laurent polynomial symbol* A *is given by (2.21) for some Laurent polynomial* B *satisfying (2.22). Then the subdivision scheme (1.18) converges for any control point sequence* $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in \ell^\infty(\mathbb{Z})$ *if and only if there is an integer* $r \in \mathbb{N}$ *such that the operator* $S_{\boldsymbol{b}}^r$ *is contractive.*

Observe from (1.45) that, for the case $A = A_1$, the subdivision mask symbol corresponding to the sequence $\{a_j\}$ given by (1.19), we have that (2.21) is satisfied by

$$B(z) = \tfrac{1}{2}z^{-1}(1+z) = \tfrac{1}{2}z^{-1} + \tfrac{1}{2}, \tag{2.23}$$

for which it follows from the formula (2.5) that

$$\|S_{\mathbf{b}}\|_\infty = \tfrac{1}{2} < 1, \tag{2.24}$$

according to which the contractivity condition of Theorem 2.3 is satisfied for $r = 1$.

We proceed to apply Theorem 2.3 to derive certain useful sufficient conditions, to be formulated in Theorems 2.4 and 2.5 below, that assure the subdivision operator $S_{\mathbf{a}}$ to provide a convergent subdivision scheme. The first result from [15, Theorem 4] (see also [17, Theorem 4.13]) shows that multiplication by the factor $\frac{1+z}{2}$ of a subdivision mask symbol preserves subdivision convergence.

THEOREM 2.4 *Let* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ *be a subdivision mask sequence such that the subdivision scheme (1.18) converges for any control point sequence* $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in \ell^\infty(\mathbb{Z})$*. Then, for the Laurent polynomial*

$$\widetilde{A}(z) := \Big(\frac{1+z}{2}\Big)A(z), \qquad z \in \mathbb{C}\setminus\{0\}, \tag{2.25}$$

and corresponding sequence $\tilde{\boldsymbol{a}} = \{\tilde{a}_j\} \in M_0(\mathbb{Z})$ *defined by*

$$\sum_j \tilde{a}_j z^j := \widetilde{A}(z), \qquad z \in \mathbb{C}\setminus\{0\}, \tag{2.26}$$

the subdivision scheme

$$c_j^{(0)} := c_j \quad ; \quad c_j^{(r+1)} := \left(S_{\tilde{a}} c^{(r)}\right)_j = \left(S_{\tilde{a}}^{r+1} c\right)_j, \tag{2.27}$$

for $j \in \mathbb{Z}$ *and* $r = 0, 1, \ldots,$ *converges for any control point sequence* $c = \{c_j\} \in \ell^\infty(\mathbb{Z})$.

Next, we apply Theorems 2.3 and 2.4, to deduce a sufficient condition for subdivision convergence on the residual polynomial B with respect to the factor $(1+z)^\mu$, for some $\mu \in \mathbb{N}$, of the symbol A. Our result is as follows.

THEOREM 2.5 *Let* $a = \{a_j\} \in M_0(\mathbb{Z})$ *be a subdivision mask sequence such that for an integer* $\mu \in \mathbb{N}$, *its corresponding Laurent polynomial symbol* A, *as defined by (1.21), is given by*

$$A(z) = (1+z)^\mu B(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{2.28}$$

where the Laurent polynomial B *satisfies the condition*

$$B(1) = \frac{1}{2^{\mu-1}}. \tag{2.29}$$

Then, if it moreover holds that

$$\|S_b\|_\infty < \frac{1}{2^{\mu-1}}, \tag{2.30}$$

with S_b *denoting the subdivision operator defined as in (2.4) in terms of the sequence* $b = \{b_j\} \in M_0(\mathbb{Z})$ *given by*

$$\sum_j b_j z^j := B(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{2.31}$$

the subdivision scheme (1.18) is convergent for any control point sequence $c = \{c_j\} \in \ell^\infty(\mathbb{Z})$.

Proof. Observe from (2.28) that

$$A(z) = \left(\frac{1+z}{2}\right)^{\mu-1} \widetilde{A}(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{2.32}$$

where

$$\widetilde{A}(z) := (1+z)\widetilde{B}(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{2.33}$$

with

$$\widetilde{B}(z) := 2^{\mu-1}B(z), \qquad z \in \mathbb{C} \setminus \{0\}. \tag{2.34}$$

It follows from (2.33), (2.34) and (2.29) that

$$\widetilde{A}(1) = 2 \qquad ; \qquad \widetilde{A}(-1) = 0,$$

which is equivalent to the sum rule

$$\sum_j \tilde{a}_{2j} = \sum_j \tilde{a}_{2j+1} = 1.$$

Also, (2.34), (2.30) and (2.5) yield, for the subdivision operator $S_{\tilde{b}}$ with respect to the sequence $\tilde{\mathbf{b}} = \{\tilde{b}_j\} \in M_0(\mathbb{Z})$ given by

$$\sum_j \tilde{b}_j z^j := \widetilde{B}(z), \quad z \in \mathbb{C} \setminus \{0\},$$

the inequality

$$\|S_{\tilde{b}}\|_\infty = 2^{\mu-1} \|S_{\mathbf{b}}\|_\infty < 1,$$

that is, $S_{\tilde{b}}$ is a contractive operator. Hence we may apply Theorem 2.3 with $r = 1$ to deduce that the subdivision scheme (2.27), with the subdivision sequence $\{\tilde{a}_j\}$ defined by (2.26), is convergent for each control point sequence $\mathbf{c} = \{\mathbf{c}_j\} \in \ell^\infty(\mathbb{Z})$, which, according to (2.32), proves our result for the case $\mu = 1$. If $\mu \geq 2$, we recall (2.32), and successively apply Theorem 2.4 altogether $\mu - 1$ times to complete our proof. ■

In the next section, we apply the result of Theorem 2.5 to the class $\mathcal{A}_{m,n}$.

2.3 Application to the class $\mathcal{A}_{m,n}$

Our main interpolatory subdivision convergence result of Chapter 3 will be formulated in terms of the following sequence of residual Laurent polynomials.

Definition 2.3.1 *For a Laurent polynomial* $A \in \mathcal{A}_{m,n}$*, the Laurent polynomial sequence* $\{B_1, \ldots, B_m\}$ *defined by*

$$B_\ell(z) := \frac{A(z)}{(1+z)^{2\ell}}, \qquad z \in \mathbb{C} \setminus \{0\}, \quad \ell = 1, \ldots, m, \tag{2.35}$$

will be called ***the residual polynomials*** *with respect to* A.

It is immediately clear from (1.26) and (2.35) that B_ℓ is indeed a Laurent polynomial for each $\ell = 1, \ldots, m$, and satisfies

$$A(z) = (1+z)^{2\ell} B_\ell(z), \qquad z \in \mathbb{C} \setminus \{0\}, \quad \ell = 1, \ldots, m. \tag{2.36}$$

Observe from (2.35) and (1.25) that

$$B_\ell(1) = \frac{1}{2^{2\ell-1}}, \qquad \ell = 1, \ldots, m. \tag{2.37}$$

Also, with the sequence $\mathbf{b}_\ell = \{b_{\ell,j} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ defined by

$$\sum_j b_{\ell,j} z^j := B_\ell(z), \qquad z \in \mathbb{C} \setminus \{0\}, \quad \ell = 1, \ldots, m, \tag{2.38}$$

we see from (2.36) and condition (b) in Theorem 1.2 that $\{b_{\ell,j}\} \in M_0(\mathbb{Z})$, with

$$\operatorname{supp}\{b_{\ell,j}\} = [-2n+1, 2n-1-2\ell]|_{\mathbb{Z}}, \qquad \ell = 1, \ldots, m, \tag{2.39}$$

and thus

$$B_\ell(z) = \sum_{j=-2n+1}^{2n-1-2\ell} b_{\ell,j} z^j, \qquad z \in \mathbb{C} \setminus \{0\}, \quad \ell = 1, \ldots, m. \tag{2.40}$$

According to (1.17) in the Definition 1.2.2, the sequence $\{S_{\mathbf{b}_1}, \ldots, S_{\mathbf{b}_m}\}$ of subdivision operators is defined by

$$\left(S_{\mathbf{b}_\ell} \mathbf{c}\right)_j := \sum_k b_{\ell,j-2k} \mathbf{c}_k, \qquad j \in \mathbb{Z}, \quad \ell = 1, \ldots, m, \tag{2.41}$$

for any control point sequence $\{\mathbf{c}_j\} \in M^s(\mathbb{Z})$.

We proceed to apply the subdivision convergence result of Theorem 2.5 to interpolatory subdivision schemes with Laurent polynomial symbols $A \in \mathcal{A}_{m,n}$, to immediately obtain the following result.

THEOREM 2.6 *Let* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ *be a subdivision mask sequence such that its corresponding Laurent polynomial symbol* A*, as given by (1.21), belongs to the class* $\mathcal{A}_{m,n}$*, and suppose there is an integer* $\ell \in \{1, \ldots, m\}$ *for which the subdivision operator* $S_{\boldsymbol{b}_\ell}$*, as defined by (2.41),(2.38) and (2.35), satisfies the condition*

$$\|S_{\boldsymbol{b}_\ell}\|_\infty < \frac{1}{2^{2\ell-1}}. \tag{2.42}$$

Then the interpolatory subdivision scheme (1.18) is convergent for any control point sequence $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in \ell^\infty(\mathbb{Z})$.

In Chapters 3 and 4, we proceed to develop an interpolatory subdivision convergence theory according to which Theorem 2.6 holds with (2.42) replaced by the weaker condition that $S_{\mathbf{b}_\ell}$ is a contractive operator, and in which the admissible set of control point sequences is extended to the whole of $M^s(\mathbb{Z})$.

Interpolatory Refinable Functions from the Cascade Algorithm

In this chapter, we introduce and apply the cascade algorithm to derive a weaker (contractive) condition than (2.42) in Theorem 2.6 on the residual Laurent polynomial subdivision operator, which together with a further restrictive condition, guarantees the existence of a corresponding interpolatory refinable function, and thus also, in view of Theorem 1.6, assures the convergence of the associated interpolatory subdivision scheme, for all control point sequences $\{\mathbf{c}_j\} \in \mathsf{M}^s(\mathbb{Z})$.

3.1 The cascade algorithm

First, we introduce the notion of cascade operators and their properties, as follows.

Definition 3.1.1 *For a sequence* $\boldsymbol{a} = \{a_j\} \in \mathsf{M}_0(\mathbb{Z})$*, the operator* $\mathsf{T}_{\boldsymbol{a}} : \mathsf{C}(\mathbb{R}) \longrightarrow \mathsf{C}(\mathbb{R})$ *defined for any* $f \in \mathsf{C}(\mathbb{R})$ *by*

$$\Big(\mathsf{T}_{\boldsymbol{a}} f\Big)(x) := \sum_j a_j f(2x - j), \qquad x \in \mathbb{R}, \tag{3.1}$$

is called the ***cascade operator****.*

Observe from (3.1) that the refinement equation (1.49) has the equivalent formulation

$$\phi = \mathsf{T}_{\mathbf{a}} \phi \tag{3.2}$$

in terms of the cascade operator.

In order to obtain a constructive existence proof of a solution $\phi \in C_0(\mathbb{R})$ of the refinement equation (1.49), it is natural to introduce the ***cascade algorithm***, which generates, for a given sequence $\mathbf{a} = \{a_j\} \in M_0(\mathbb{Z})$ and an initial function $g \in C(\mathbb{R})$, the sequence $\{\phi_r : r = 0, 1, \ldots\} \subset C(\mathbb{R})$ by means of

$$\phi_0 := g \quad ; \quad \phi_{r+1} := T_a\phi_r = T_a^{r+1}g, \quad r = 0, 1, \ldots. \tag{3.3}$$

DEFINITION 3.1.2 *The symbol* $C_u(\mathbb{R})$ *denotes the subspace of* $C(\mathbb{R})$ *consisting of uniformly bounded continuous functions on* $\mathbb{R}$*, that is,* $f \in C_u(\mathbb{R})$ *if and only if* $\|f\|_\infty := \sup\limits_x |f(x)| = \sup\limits_{x\in\mathbb{R}} |f(x)| < \infty$.

Note that we use the same symbol $\|\cdot\|_\infty$ for uniformly bounded sequences in $M(\mathbb{Z})$, and functions in $C(\mathbb{R})$. The intended meaning will always be clear from the context. Also, observe that $C_0(\mathbb{R}) \subset C_u(\mathbb{R})$.

The following properties of the cascade operator are of fundamental importance to our next discussions.

THEOREM 3.1 *For a sequence* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$*, the cascade operator* $T_{\boldsymbol{a}} : C(\mathbb{R}) \longrightarrow C(\mathbb{R})$*, as defined by (3.1), satisfies*

(a) *if* $f \in C_u(\mathbb{R})$*, then* $T_{\boldsymbol{a}}f \in C_u(\mathbb{R})$*, with*

$$\|T_{\boldsymbol{a}}f\|_\infty \leq \Big[\sum_j |a_j|\Big] \|f\|_\infty ; \tag{3.4}$$

(b) *if* $\mathrm{supp}\{a_j\} = [\mu, \nu]|_{\mathbb{Z}}$*, and* $f \in C_0(\mathbb{R})$*, with* $\mathrm{supp}^c f = [\alpha, \beta]$*, then* $T_{\boldsymbol{a}}f \in C_0(\mathbb{R})$*, with*

$$\mathrm{supp}^c\big(T_{\boldsymbol{a}}f\big) = \Big[\tfrac{1}{2}(\mu + \alpha), \tfrac{1}{2}(\nu + \beta)\Big]; \tag{3.5}$$

(c) *if* $\{a_j\}$ *satisfies the sum rules*

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1, \tag{3.6}$$

and $f \in C_0(\mathbb{R})$*, with*

$$\sum_j f(x - j) = 1, \qquad x \in \mathbb{R}, \tag{3.7}$$

then

$$\sum_j (T_a f)(x-j) = 1, \qquad x \in \mathbb{R}; \tag{3.8}$$

(d) *if* $\{a_j\}$ *satisfies the condition (1.10), and* $f \in C_u(\mathbb{R})$, *with*

$$f(j) = \delta_j, \qquad j \in \mathbb{Z}, \tag{3.9}$$

then

$$(T_a f)(j) = \delta_j, \qquad j \in \mathbb{Z}; \tag{3.10}$$

(e) *for any* $f \in C_0(\mathbb{R})$ *and with* S_a *denoting the subdivision operator defined by (1.17),*

$$(T_a^r f)(x) = \sum_j \left(S_a^r \delta\right)_j f(2^r x - j), \qquad x \in \mathbb{R}, \quad r = 1, 2, \ldots, \tag{3.11}$$

where $\delta = \{\delta_j\}$*, the delta sequence.*

Proof. (a) Let $f \in C_u(\mathbb{R})$. Then, from (3.1), and for any $x \in \mathbb{R}$,

$$|(T_a f)(x)| \leq \sum_j |a_j|\,|f(2x-j)| \leq \left[\sum_j |a_j|\right] \|f\|_\infty ,$$

from which it then immediately follows that $T_a f \in C_u(\mathbb{R})$, and that (3.4) holds.

(b) Suppose $\mathrm{supp}\{a_j\} = [\mu, \nu]|_{\mathbb{Z}}$, and $f \in C_0(\mathbb{R})$, with $\mathrm{supp}^c f = [\alpha, \beta]$. Then (3.1) gives, for any $x \in \mathbb{R}$,

$$(T_a f)(x) = \sum_{j=\mu}^{\nu} a_j f(2x-j),$$

and thus, since $a_\mu \neq 0$ and $2x - \mu = \alpha$ if and only if $x = \frac{1}{2}(\mu + \alpha)$, whereas also $a_\nu \neq 0$ and $2x - \nu = \beta$ if and only if $x = \frac{1}{2}(\mu + \beta)$, it follows that (3.5) is satisfied.

(c) Let $\{a_j\}$ satisfy the sum rules (3.6), and suppose $f \in C_0(\mathbb{R})$ is such that (3.7) holds. From (3.1) we obtain, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_j (T_a f)(x-j) &= \sum_j \sum_k a_k f(2x - 2j - k) \\
&= \sum_j \sum_k a_{k-2j} f(2x-k) \\
&= \sum_k \left[\sum_j a_{k-2j}\right] f(2x-k).
\end{aligned} \tag{3.12}$$

Since, moreover, (3.6) yields

$$\sum_k a_{2j-2k} = \sum_k a_{2k} = 1, \quad j \in \mathbb{Z},$$

and

$$\sum_k a_{2j+1-2k} = \sum_k a_{2k+1} = 1, \quad j \in \mathbb{Z},$$

we deduce that

$$\sum_k a_{j-2k} = 1, \quad j \in \mathbb{Z}. \tag{3.13}$$

By substituting (3.13) into (3.12) we obtain, for any $x \in \mathbb{R}$,

$$\sum_j (T_{\mathbf{a}}f)(x-j) = \sum_k f(2x-k) = 1,$$

from (3.7), and thereby completing our proof of (3.8).

(d) Let $\{a_j\}$ satisfy (1.10), and suppose $f \in C_u(\mathbb{R})$ satisfies (3.9). Then (3.1) gives, for any $j \in \mathbb{Z}$,

$$\begin{aligned}(T_{\mathbf{a}}f)(j) = \sum_k a_k f(2j-k) &= \sum_k a_{2j-k} f(k) \\ &= \sum_k a_{2j-k}\delta_k = a_{2j} = \delta_j,\end{aligned}$$

which proves (3.10).

(e) Let $f \in C_0(\mathbb{R})$. Since (1.17) gives

$$\left(S_{\mathbf{a}}\delta\right)_j = \sum_k a_{j-2k}\delta_k = a_j, \qquad j \in \mathbb{Z}, \tag{3.14}$$

we see from (3.1) that (3.11) holds for $r = 1$. Suppose next that the integer $r \geq 2$. Then, from (1.17), (3.1) and (3.14), and for any $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_j (S_{\mathbf{a}}^r\delta)_j f(2^r x - j) &= \sum_j \Big(S_{\mathbf{a}}(S_{\mathbf{a}}^{r-1}\delta)\Big)_j f(2^r x - j) \\
&= \sum_j \Big[\sum_k a_{j-2k}(S_{\mathbf{a}}^{r-1}\delta)_k\Big] f(2^r x - j) \\
&= \sum_k (S_{\mathbf{a}}^{r-1}\delta)_k \Big[\sum_j a_{j-2k} f(2^r x - j)\Big] \\
&= \sum_k (S_{\mathbf{a}}^{r-1}\delta)_k \Big[\sum_j a_j f\big(2(2^{r-1}x - k) - j\big)\Big] \\
&= \sum_k (S_{\mathbf{a}}^{r-1}\delta)_k (T_{\mathbf{a}} f)(2^{r-1}x - k) \\
&= \cdots \\
&= \sum_k (S_{\mathbf{a}}\delta)_k (T_{\mathbf{a}}^{r-1} f)(2x - k) \\
&= \sum_k a_k (T_{\mathbf{a}}^{r-1} f)(2x - k) = \Big(T_{\mathbf{a}}(T_{\mathbf{a}}^{r-1} f)\Big)(x) \\
&= (T_{\mathbf{a}}^r f)(x),
\end{aligned}$$

which completes our proof of (3.11). ■

Theorem 3.1 has the following implications with respect to the cascade algorithm.

THEOREM 3.2 *For a sequence* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$, *with*

$$\operatorname{supp}\{a_j\} = [\mu, \nu]|_{\mathbb{Z}}\,, \tag{3.15}$$

let $\{\phi_r : r = 0, 1, \ldots\} \subset C(\mathbb{R})$ *denote the sequence of functions generated by the cascade algorithm (3.3) with initial function* $g \in C_0(\mathbb{R})$. *Then*

(a) *if*

$$\operatorname{supp}^c g = [\alpha, \beta], \tag{3.16}$$

it holds, for $r = 0, 1, \ldots$, *that*

$$\phi_r \in C_0(\mathbb{R}), \tag{3.17}$$

with

$$\operatorname{supp}^c \phi_r = \Big[\mu + \frac{\alpha - \mu}{2^r}, \nu - \frac{\nu - \beta}{2^r}\Big]; \tag{3.18}$$

(b) *if the sequence* $\{a_j\}$ *satisfies the property (1.10), and*

$$g(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{3.19}$$

it holds, for $r = 0, 1, \ldots$, *that*

$$\phi_r(j) = \delta_j, \quad j \in \mathbb{Z}. \tag{3.20}$$

Proof. (a) Since (3.3) gives $\phi_0 := g$, it follows from (3.16) that (3.18) holds for $r = 0$. Proceeding inductively, we suppose next that (3.18) is satisfied for a fixed integer $r \in \{0, 1, \ldots\}$. Since (3.3) gives

$$\phi_{r+1} = T_{\mathbf{a}}\phi_r, \tag{3.21}$$

and $a_\nu \neq 0$, we may deduce from the inductive hypothesis (3.18), together with Theorem 3.1(b),that

$$\text{supp}^c\phi_{r+1} = [\alpha_r, \beta_r], \tag{3.22}$$

where

$$\alpha_r := \frac{\left(\mu + \frac{\alpha-\mu}{2^r}\right) + \mu}{2} = \mu + \frac{\alpha - \mu}{2^{r+1}}, \tag{3.23}$$

and similarly

$$\beta_r := \frac{\left(\nu - \frac{\nu-\beta}{2^r}\right) + \nu}{2} = \nu - \frac{\nu - \beta}{2^{r+1}}. \tag{3.24}$$

According to (3.22),(3.23) and (3.24), the support property (3.18) is also satisfied with r replaced by $r+1$, which completes the inductive proof of (3.18).

(b) First, since $\phi_0 := g$, we see from (3.19) that (3.20) holds for $r = 0$. Proceeding inductively, suppose next that (3.20) is satisfied for an integer $r \in \{0, 1, \ldots\}$. It follows from (3.3), (3.1), the inductive hypothesis (3.20), (1.10), and Theorem 3.1(d), that

$$\phi_{r+1}(j) = (T_a\phi_r)(j) = \delta_j, \quad j \in \mathbb{Z},$$

which shows that (3.20) is also satisfied with r replaced by $r+1$, and thereby completing our inductive proof of (3.20). ■

In Section 3.3, we shall use Theorem 3.1 to show that, for a sequence $\{a_j\}$ with corresponding Laurent polynomial symbol $A \in \mathcal{A}_{m,n}$, the cascade algorithm (3.3), with a suitable choice of initial function g, and subject to a certain contractivity condition, converges uniformly on $\mathbb{R}$ to an interpolatory refinable function ϕ with refinement sequence $\{a_j\}$.

3.2 The difference operator

The following definition and lemmas are going to be necessary in Section 3.3.

Definition 3.2.1 *The* ***difference operator*** $\Delta : M^s(\mathbb{Z}) \longrightarrow M^s(\mathbb{Z})$ *is defined by*

$$(\Delta \boldsymbol{c})_j := \boldsymbol{c}_j - \boldsymbol{c}_{j-1}, \quad j \in \mathbb{Z}, \tag{3.25}$$

for any sequence $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in M^s(\mathbb{Z})$.

Also, we recursively define

$$\Delta^{k+1} := \Delta(\Delta^k), \quad k = 0, 1, \ldots, \tag{3.26}$$

where $\Delta^0 \mathbf{c} := \mathbf{c}$, for any $\mathbf{c} \in M^s(\mathbb{Z})$.

An inductive proof based on (3.25) and (3.26) yields, for any integer $k \in \mathbb{N}$, the explicit formula

$$(\Delta^k \mathbf{c})_j = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \mathbf{c}_{j-\ell}, \quad j \in \mathbb{Z}, \tag{3.27}$$

for each sequence $\mathbf{c} = \{\mathbf{c}_j\} \in M^s(\mathbb{Z})$.

Our cascade algorithm convergence result in Section 3.3 will rely on the following sequence of lemmas involving the difference operator.

Lemma 3.3 *For a sequence* $\boldsymbol{c} = \{c_j\} \in M_0(\mathbb{Z})$ *and an integer* $\ell \in \mathbb{N}$, *let the Laurent polynomials* C *and* D_ℓ *be defined by*

$$C(z) := \sum_j c_j z^j, \quad z \in \mathbb{C} \setminus \{0\}; \tag{3.28}$$

$$D_\ell(z) := \sum_j (\Delta^\ell c)_j z^j, \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.29}$$

Then

$$D_\ell(z) = (1-z)^\ell C(z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.30}$$

Proof. By using (3.29), (3.27) and (3.28), we obtain, for any sequence $\mathbf{c} = \{c_j\} \in M_0(\mathbb{Z})$, integer $\ell \in \mathbb{N}$, and $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} D_\ell(z) &= \sum_j \Big[\sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} c_{j-k}\Big] z^j \\ &= \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \Big[\sum_j c_{j-k} z^{j-k}\Big] z^k \\ &= \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \Big[\sum_j c_j z^j\Big] z^k \\ &= C(z)\Big[\sum_{k=0}^{\ell} \binom{\ell}{k} (-z)^k\Big] \\ &= C(z)(1-z)^\ell, \end{aligned}$$

and thereby proving (3.30). ■

Lemma 3.4 *For two sequences $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ and $\boldsymbol{c} = \{c_j\} \in M_0(\mathbb{Z})$, let the Laurent polynomials A and C be defined by (1.21) and (3.28). Then the Laurent polynomial E defined by*

$$E(z) := \sum_j (S_{\boldsymbol{a}} c)_j z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{3.31}$$

with $S_{\boldsymbol{a}}$ denoting the subdivision operator given by (1.17), satisfies the identity

$$E(z) = A(z)C(z^2), \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.32}$$

Proof. It follows from (3.31),(1.17), (1.21), and (3.28), that for any $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} E(z) &= \sum_j \Big[\sum_k a_{j-2k} c_k \Big] z^j \\ &= \sum_k c_k \Big[\sum_j a_{j-2k} z^{j-2k} \Big] z^{2k} \\ &= \sum_k c_k \Big[\sum_j a_j z^j \Big] z^{2k} \\ &= A(z) \sum_k c_k (z^2)^k = A(z) C(z^2), \end{aligned}$$

which proves (3.32). ■

LEMMA 3.5 *Let $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ be a sequence with Laurent polynomial symbol $A \in \mathcal{A}_{m,n}$, and for an integer $\ell \in \{1, \dots, m\}$, let the sequence $\boldsymbol{b}_\ell = \{b_{\ell,j}\} \in M_0(\mathbb{Z})$ be defined by (2.38) and (2.35). Then the subdivision operators $S_{\boldsymbol{a}}$ and $S_{\boldsymbol{b}_\ell}$, as defined by (1.17) and (2.41), satisfy*

$$\Delta^{2\ell}(S_{\boldsymbol{a}}^r \boldsymbol{c}) = S_{\boldsymbol{b}_\ell}^r (\Delta^{2\ell} \boldsymbol{c}), \quad r = 1, 2, \dots, \tag{3.33}$$

for any sequence $\boldsymbol{c} \in M_0(\mathbb{Z})$.

Proof. First, we shall establish that (3.33) holds for $r = 1$, that is,

$$\Delta^{2\ell}(S_{\mathbf{a}} \mathbf{c}) = S_{\mathbf{b}_\ell}(\Delta^{2\ell} \mathbf{c}), \tag{3.34}$$

for any sequence $\mathbf{c} = \{c_j\} \in M_0(\mathbb{Z})$. To this end, we use Lemmas 3.3 and 3.4, together with (2.36), to obtain, for any $\mathbf{c} = \{c_j\} \in M_0(\mathbb{Z})$, and $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
\sum_j \Big(\Delta^{2\ell}(S_{\mathbf{a}}\mathbf{c})\Big)_j z^j &= (1-z)^{2\ell}\Big[\sum_j (S_{\mathbf{a}}\mathbf{c})_j z^j\Big] \\
&= (1-z)^{2\ell}A(z)C(z^2) \\
&= (1-z)^{2\ell}\Big[(1+z)^{2\ell}B_\ell(z)\Big]C(z^2) \\
&= B_\ell(z)(1-z^2)^{2\ell}C(z^2) \\
&= B_\ell(z)\Big[\sum_j (\Delta^{2\ell}\mathbf{c})_j (z^2)^j\Big] \\
&= \sum_j \Big(S_{\mathbf{b}_\ell}(\Delta^{2\ell}\mathbf{c})\Big)_j z^j,
\end{aligned}$$

and thus

$$\Big(\Delta^{2\ell}(S_{\mathbf{a}}\mathbf{c})\Big)_j = \Big(S_{\mathbf{b}_\ell}(\Delta^{2\ell}\mathbf{c})\Big)_j, \qquad j \in \mathbb{Z},$$

which yields the desired result (3.34). Suppose next $r \geq 2$, and let $\mathbf{c} = \{c_j\} \in M_0(\mathbb{Z})$. Repeated application of (3.34) then yields

$$\begin{aligned}
\Delta^{2\ell}(S_{\mathbf{a}}^r\mathbf{c}) &= \Delta^{2\ell}\Big(S_{\mathbf{a}}(S_{\mathbf{a}}^{r-1}\mathbf{c})\Big) \\
&= S_{\mathbf{b}_\ell}\Big(\Delta^{2\ell}(S_{\mathbf{a}}^{r-1}\mathbf{c})\Big) \\
&= \ldots \\
&= S_{\mathbf{b}_\ell}^r(\Delta^{2\ell}\mathbf{c}),
\end{aligned}$$

which is the required result (3.33). ■

Lemma 3.6 *Let the sequences $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ and $\boldsymbol{b}_\ell = \{b_{\ell,j}\} \in M_0(\mathbb{Z})$ be as in Lemma 3.5, for any integer $\ell \in \{1, \ldots, m\}$. Then, for any sequence $\boldsymbol{c} = \{c_j\} \in M(\mathbb{Z})$,*

$$c_j - \sum_k a_{2k+1}c_{j-2k-1} = (\Delta^{2\ell}\boldsymbol{\gamma}_\ell)_j, \qquad j \in \mathbb{Z}, \tag{3.35}$$

with the sequence $\boldsymbol{\gamma}_\ell = \{\gamma_{\ell,j}\} \in M_0(\mathbb{Z})$ defined by

$$\gamma_{\ell,j} := \sum_k (-1)^k b_{\ell,k}c_{j-k}, \qquad j \in \mathbb{Z}. \tag{3.36}$$

Proof. First, we use the fact that, from Theorem 1.2, we have $a_{2j} = \delta_j, j \in \mathbb{Z}$, to obtain, with the Laurent polynomials A and C defined by (1.21) and (3.28), and for $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
\sum_j \Big(c_j - \sum_k a_{2k+1} c_{j-2k-1} \Big) z^j &= \sum_j \Big[\sum_k (-1)^k a_k c_{j-k} \Big] z^j \\
&= \sum_k (-1)^k a_k \Big[\sum_j c_{j-k} z^{j-k} \Big] z^k \\
&= \sum_k (-1)^k a_k \Big[\sum_j c_j z^j \Big] z^k \\
&= C(z) \sum_k a_k (-z)^k = C(z) A(-z). \qquad (3.37)
\end{aligned}$$

Next, we use Lemma 3.3, together with (3.36), (2.38) and (2.36), to deduce that, for any $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
\sum_j (\Delta^{2\ell} \boldsymbol{\gamma}_\ell)_j z^j &= (1-z)^{2\ell} \Big[\sum_j \gamma_{\ell,j} z^j \Big] \\
&= (1-z)^{2\ell} \sum_j \Big[\sum_k (-1)^k b_{\ell,k} c_{j-k} \Big] z^j \\
&= (1-z)^{2\ell} \sum_k (-1)^k b_{\ell,k} \Big[\sum_j c_{j-k} z^{j-k} \Big] z^k \\
&= (1-z)^{2\ell} \sum_k (-1)^k b_{\ell,k} \Big[\sum_j c_j z^j \Big] z^k \\
&= (1-z)^{2\ell} C(z) \sum_k b_{\ell,k} (-z)^k \\
&= (1-z)^{2\ell} C(z) B_\ell(-z) \\
&= C(z) \Big[(1-z)^{2\ell} B_\ell(-z) \Big] = C(z) A(-z). \qquad (3.38)
\end{aligned}$$

It follows from (3.37) and (3.38) that

$$\sum_j \Big(c_j - \sum_k a_{2k+1} c_{j-2k-1} \Big) z^j = \sum_j (\Delta^{2\ell} \boldsymbol{\gamma}_\ell)_j z^j, \quad z \in \mathbb{C} \setminus \{0\},$$

from which the desired result (3.35) then immediately follows. ■

3.3 Cascade algorithm convergence

We proceed to apply the preliminary results in Sections 3.1 and 3.2 to establish our following main convergence and existence result of this chapter.

THEOREM 3.7 *Let* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ *denote a sequence for which the corresponding Laurent polynomial symbol* A*, as defined by (1.21), satisfies* $A \in \mathcal{A}_{m,n}$*, and suppose there is an integer* $\ell \in \{1, \ldots, m\}$ *such that the following two conditions are satisfied:*

(a) *there exists an interpolatory refinable function* $\widetilde{\phi}$ *with refinement sequence* $\tilde{\boldsymbol{a}} = \{\tilde{a}_j\}$ *for which the corresponding Laurent polynomial symbol*

$$\widetilde{A}(z) := \sum_j \tilde{a}_j z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{3.39}$$

satisfies $\widetilde{A} \in \mathcal{A}_{\ell,\tilde{n}}$ *for some integer* $\tilde{n} \in \{\ell, \ldots, n\}$*;*

(b) *the subdivision operator* $S_{\boldsymbol{b}_\ell}$ *defined by (2.41), (2.38) and (2.35) is contractive, that is,*

$$\|S_{\boldsymbol{b}_\ell}\|_\infty < 1. \tag{3.40}$$

Then there exists a function $\phi \in C_0(\mathbb{R})$ *such that the cascade algorithm (3.3) with initial function* $\widetilde{\phi}$ *converges uniformly on* $\mathbb{R}$ *to* ϕ*, that is,*

$$\left\| \phi - T_{\boldsymbol{a}}^r \widetilde{\phi} \right\|_\infty \longrightarrow 0, \qquad r \longrightarrow \infty. \tag{3.41}$$

Moreover, ϕ *is an interpolatory refinable function with refinement sequence* $\{a_j\}$*.*

Remark. Observe that the condition (a) of Theorem 3.7 is satisfied for $\ell = 1$ by $\widetilde{\phi} = h$, the hat function, for which $\widetilde{A} = A_1$, as given by (1.45), so that, according to (1.46), we have $\widetilde{A} \in \mathcal{A}_{1,1}$, and thus $\tilde{n} = 1$.

Proof of Theorem 3.7. Let $r \in \{0, 1, \ldots\}$, and fix $x \in \mathbb{R}$. By using (3.3) with $g = \widetilde{\phi}$, and applying (3.11) in Theorem 3.1(e), as well as (1.18), we obtain

$$\begin{aligned} \phi_{r+1}(x) - \phi_r(x) &= (T_{\mathbf{a}}^{r+1}\widetilde{\phi})(x) - (T_{\mathbf{a}}^r\widetilde{\phi})(x) \\ &= \sum_j c_j^{(r+1)} \widetilde{\phi}(2^{r+1}x - j) - \sum_j c_j^{(r)} \widetilde{\phi}(2^r x - j), \end{aligned} \tag{3.42}$$

where

$$c_j^{(r)} := (S_{\mathbf{a}}^r \boldsymbol{\delta})_j, \qquad j \in \mathbb{Z}, \tag{3.43}$$

with $\boldsymbol{\delta} = \{\delta_j\}$ denoting the delta sequence. Now apply (1.18) and (1.17) to obtain

$$\sum_j c_j^{(r+1)} \widetilde{\phi}(2^{r+1}x - j) = \sum_j \Big[\sum_k a_{j-2k} c_k^{(r)} \Big] \widetilde{\phi}(2^{r+1}x - j). \tag{3.44}$$

Next, we use the refinability of $\widetilde{\phi}$ with respect to the sequence $\{\tilde{a}_j\}$ to deduce that

$$\begin{aligned} \sum_j c_j^{(r)} \widetilde{\phi}(2^r x - j) &= \sum_j c_j^{(r)} \sum_k \tilde{a}_k \widetilde{\phi}(2^{r+1}x - 2j - k) \\ &= \sum_j c_j^{(r)} \sum_k \tilde{a}_{k-2j} \widetilde{\phi}(2^{r+1}x - k) \\ &= \sum_k \Big[\sum_j \tilde{a}_{k-2j} c_j^{(r)} \Big] \widetilde{\phi}(2^{r+1}x - k) \\ &= \sum_j \Big[\sum_k \tilde{a}_{j-2k} c_k^{(r)} \Big] \widetilde{\phi}(2^{r+1}x - j). \end{aligned} \tag{3.45}$$

By substituting (3.44) and (3.45) into (3.42), we obtain

$$\phi_{r+1}(x) - \phi_r(x) = \sum_j \Big[\sum_k (a_{j-2k} - \tilde{a}_{j-2k}) c_k^{(r)} \Big] \widetilde{\phi}(2^{r+1}x - j). \tag{3.46}$$

Since $A \in \mathcal{A}_{m,n}$ and $\widetilde{A} \in \mathcal{A}_{\ell,\tilde{n}}$, it follows from Theorem 1.2 that

$$a_{2j} = \delta_j, \quad j \in \mathbb{Z} \qquad ; \qquad \tilde{a}_{2j} = \delta_j, \quad j \in \mathbb{Z}, \tag{3.47}$$

which we now use in (3.46) to deduce that

$$\begin{aligned} \phi_{r+1}(x) - \phi_r(x) &= \sum_j \Big[\sum_k (a_{2j-2k} - \tilde{a}_{2j-2k}) c_k^{(r)} \Big] \widetilde{\phi}(2^{r+1}x - 2j) \\ &\quad + \sum_j \Big[\sum_k (a_{2j+1-2k} - \tilde{a}_{2j+1-2k}) c_k^{(r)} \Big] \widetilde{\phi}(2^{r+1}x - 2j - 1) \\ &= \sum_j \Big[\sum_k (a_{2j+1-2k} - \tilde{a}_{2j+1-2k}) c_k^{(r)} \Big] \widetilde{\phi}(2^{r+1}x - 2j - 1) \\ &= \sum_j \Big[\sum_k a_{2j+1-2k} c_k^{(r)} - \sum_k \tilde{a}_{2j+1-2k} c_k^{(r)} \Big] \widetilde{\phi}(2^{r+1}x - 2j - 1) \\ &= \sum_j \Big[c_{2j+1}^{(r+1)} - \sum_k \tilde{a}_{2j+1-2k} c_{2k}^{(r+1)} \Big] \widetilde{\phi}(2^{r+1}x - 2j - 1), \end{aligned} \tag{3.48}$$

from (1.18), and the fact that $c_{2k}^{(r+1)} = c_k^{(r)}$, $k \in \mathbb{Z}$, since (1.18) is an interpolatory subdivision scheme as in (1.1). Since $\widetilde{A} \in \mathcal{A}_{\ell,\tilde{n}}$, we know from Theorem 1.2(d) that

$$\widetilde{A}^{(k)}(-1) = 0, \qquad k = 0, \ldots, 2\ell - 1,$$

and thus we may define the Laurent polynomial $\widetilde{B}$ by

$$\widetilde{B}(z) := \frac{\widetilde{A}(z)}{(1+z)^{2\ell}}, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{3.49}$$

in terms of which the sequence $\tilde{\mathbf{b}} = \{\tilde{b}_j\} \in M_0(\mathbb{Z})$ is then defined by

$$\sum_j \tilde{b}_j z^j := \widetilde{B}(z), \qquad z \in \mathbb{C} \setminus \{0\}. \tag{3.50}$$

Hence we may apply Lemma 3.6 to obtain, for any $j \in \mathbb{Z}$,

$$\begin{aligned} c_{2j+1}^{(r+1)} - \sum_k \tilde{a}_{2j+1-2k} c_{2k}^{(r+1)} &= c_{2j+1}^{(r+1)} - \sum_k \tilde{a}_{2k+1} c_{2j-2k}^{(r+1)} \\ &= c_{2j+1}^{(r+1)} - \sum_k \tilde{a}_{2k+1} c_{2j+1-2k-1}^{(r+1)} \\ &= \left(\Delta^{2\ell} \widetilde{\boldsymbol{\gamma}}^{(r)}\right)_{2j+1}, \end{aligned} \tag{3.51}$$

where $\widetilde{\boldsymbol{\gamma}}^{(r)} = \{\widetilde{\gamma}_j^{(r)}\} \in M_0(\mathbb{Z})$ is defined by

$$\widetilde{\gamma}_j^{(r)} := \sum_k (-1)^k \tilde{b}_k c_{j-k}^{(r+1)}, \qquad j \in \mathbb{Z}. \tag{3.52}$$

Now substitute (3.51) into (3.48) to obtain

$$\phi_{r+1}(x) - \phi_r(x) = \sum_j \left(\Delta^{2\ell} \widetilde{\boldsymbol{\gamma}}^{(r)}\right)_{2j+1} \widetilde{\phi}(2^{r+1}x - 2j - 1). \tag{3.53}$$

Since $\widetilde{A} \in \mathcal{A}_{\ell,\tilde{n}}$, and $\widetilde{\phi}$ is an interpolatory refinable function with refinement sequence $\{\tilde{a}_j\}$, we know from Theorem 1.7(a) that

$$\mathrm{supp}^c \widetilde{\phi} = [-2\tilde{n} + 1, 2\tilde{n} - 1]. \tag{3.54}$$

Denote by k the integer such that

$$\frac{k}{2^r} \leq x < \frac{k+1}{2^r}. \tag{3.55}$$

It then follows from (3.54) and (3.55) that

$$\sum_j \left(\Delta^{2\ell} \widetilde{\boldsymbol{\gamma}}^{(r)}\right)_{2j+1} \widetilde{\phi}(2^{r+1}x - 2j - 1) = \sum_{j=k-\tilde{n}+1}^{k+\tilde{n}-1} \left(\Delta^{2\ell} \widetilde{\boldsymbol{\gamma}}^{(r)}\right)_{2j+1} \widetilde{\phi}(2^{r+1}x - 2j - 1). \tag{3.56}$$

By using (3.56) in (3.53), we deduce that

$$\begin{aligned}|\phi_{r+1}(x)-\phi_r(x)| &\leq \sum_{j=k-\tilde{n}+1}^{k+\tilde{n}-1}\left|\left(\Delta^{2\ell}\widetilde{\boldsymbol{\gamma}}^{(r)}\right)_j\right|\left|\widetilde{\phi}(2^{r+1}x-2j-1)\right| \\ &\leq (2\tilde{n}-1)\left\|\widetilde{\phi}\right\|_\infty\left\|\Delta^{2\ell}\widetilde{\boldsymbol{\gamma}}^{(r)}\right\|_\infty. \end{aligned} \tag{3.57}$$

Next, we apply (3.27) and (3.52) to obtain, for any $j \in \mathbb{Z}$,

$$\begin{aligned}\left(\Delta^{2\ell}\widetilde{\boldsymbol{\gamma}}^{(r)}\right)_j &= \sum_{k=0}^{2\ell}(-1)^k\binom{2\ell}{k}\widetilde{\gamma}_{j-k}^{(r)} \\ &= \sum_{k=0}^{2\ell}(-1)^k\binom{2\ell}{k}\sum_i(-1)^i\tilde{b}_i c_{j-k-i}^{(r+1)} \\ &= \sum_i(-1)^i\tilde{b}_i\sum_{k=0}^{2\ell}(-1)^k\binom{2\ell}{k}c_{j-i-k}^{(r+1)} \\ &= \sum_i(-1)^i\tilde{b}_i\left(\Delta^{2\ell}\mathbf{c}^{(r+1)}\right)_{j-i},\end{aligned}$$

and thus

$$\left|\left(\Delta^{2\ell}\widetilde{\boldsymbol{\gamma}}^{(r)}\right)_j\right| \leq \|\Delta^{2\ell}\mathbf{c}^{(r+1)}\|_\infty\left(\sum_i\left|\tilde{b}_i\right|\right),$$

so that

$$\left\|\Delta^{2\ell}\widetilde{\boldsymbol{\gamma}}^{(r)}\right\|_\infty \leq \left[\sum_i\left|\tilde{b}_i\right|\right]\|\Delta^{2\ell}\mathbf{c}^{(r+1)}\|_\infty. \tag{3.58}$$

By substituting (3.58) into (3.57), we obtain

$$|\phi_{r+1}(x)-\phi_r(x)| \leq (2\tilde{n}-1)\left[\sum_i\left|\tilde{b}_i\right|\right]\left\|\widetilde{\phi}\right\|_\infty\|\Delta^{2\ell}\mathbf{c}^{(r+1)}\|_\infty. \tag{3.59}$$

Now use (3.43) and Lemma 3.5 to obtain

$$\begin{aligned}\Delta^{2\ell}\mathbf{c}^{(r+1)} &= \Delta^{2\ell}(S_{\mathbf{a}}^{r+1}\boldsymbol{\delta}) \\ &= S_{\mathbf{b}_\ell}^{r+1}(\Delta^{2\ell}\boldsymbol{\delta}),\end{aligned}$$

and thus, from (2.5), with the definition

$$\rho_\ell := \|S_{\mathbf{b}_\ell}\|_\infty = \rho_{\mathbf{b}_\ell}, \tag{3.60}$$

we have, by using also (2.6) in Theorem 2.1,

$$\begin{aligned}
\left\|\Delta^{2\ell}\mathbf{c}^{(r+1)}\right\|_\infty &= \left\|S_{\mathbf{b}_\ell}^{r+1}(\Delta^{2\ell}\boldsymbol{\delta})\right\|_\infty \\
&= \left\|S_{\mathbf{b}_\ell}\left(S_{\mathbf{b}_\ell}^{r}(\Delta^{2\ell}\boldsymbol{\delta})\right)\right\|_\infty \\
&\leq \rho_\ell \left\|S_{\mathbf{b}_\ell}^{r}(\Delta^{2\ell}\boldsymbol{\delta})\right\|_\infty \\
&\leq \cdots \\
&\leq \rho_\ell^{r+1}\left\|\Delta^{2\ell}\boldsymbol{\delta}\right\|_\infty .
\end{aligned} \tag{3.61}$$

Since, moreover, (3.27) yields, for any $j \in \mathbb{Z}$,

$$(\Delta^{2\ell}\boldsymbol{\delta})_j = \sum_{k=0}^{2\ell}(-1)^k\binom{2\ell}{k}\delta_{j-k} = (-1)^j\binom{2\ell}{j},$$

and thus

$$\left\|\Delta^{2\ell}\boldsymbol{\delta}\right\|_\infty = \max\left\{\binom{2\ell}{k} : k = 0,\ldots,2\ell\right\} = \binom{2\ell}{\ell}, \tag{3.62}$$

we may now use (3.59), (3.61) and (3.62) to obtain

$$|\phi_{r+1}(x) - \phi_r(x)| \leq (2\tilde{n}-1)\binom{2\ell}{\ell}\left[\sum_i \left|\tilde{b}_i\right|\right]\left\|\widetilde{\phi}\right\|_\infty \rho_\ell^{r+1}. \tag{3.63}$$

The right-hand-side of (3.63) is independent of x, and thus

$$\|\phi_{r+1} - \phi_r\|_\infty \leq K\rho_\ell^{r+1}, \tag{3.64}$$

where

$$K := (2\tilde{n}-1)\binom{2\ell}{\ell}\left[\sum_i \left|\tilde{b}_i\right|\right]\left\|\widetilde{\phi}\right\|_\infty . \tag{3.65}$$

Let r and s be positive integers with $s > r$. It follows from (3.64) that

$$\begin{aligned}
\|\phi_s - \phi_r\|_\infty &= \left\| \sum_{j=r}^{s-1} (\phi_{j+1} - \phi_j) \right\|_\infty \\
&\leq \sum_{j=r}^{s-1} \|\phi_{j+1} - \phi_j\|_\infty \\
&\leq K \sum_{j=r}^{s-1} \rho_\ell^{j+1} \\
&= K\rho_\ell^{r+1} \sum_{j=r}^{s-1} \rho_\ell^{j-r} \\
&= K\rho_\ell^{r+1} \sum_{j=0}^{s-r-1} \rho_\ell^{j} \\
&= K\rho_\ell^{r+1} \left[\frac{1-\rho_\ell^{s-r}}{1-\rho_\ell} \right] < \frac{K}{1-\rho_\ell} \rho_\ell^{r+1},
\end{aligned} \tag{3.66}$$

since (3.60) and (3.40) imply

$$0 < \rho_\ell < 1. \tag{3.67}$$

Let $\varepsilon > 0$. It follows from (3.66) and (3.67) that there is a positive integer $R(\varepsilon)$ such that

$$\|\phi_s - \phi_r\|_\infty < \varepsilon, \quad \text{for all} \quad s > r \geq R(\varepsilon). \tag{3.68}$$

Since $\widetilde{A} \in \mathcal{A}_{\ell,\tilde{n}}$, with $\tilde{n} \leq n$, it follows from Theorem 1.2 that

$$\operatorname{supp}\{\tilde{a}_j\} = [-2\tilde{n}+1, 2\tilde{n}-1]|_{\mathbb{Z}}, \tag{3.69}$$

and thus, from Theorem 1.7(a),

$$\operatorname{supp}^c \widetilde{\phi} = [-2\tilde{n}+1, 2\tilde{n}-1]. \tag{3.70}$$

Since also we have here $g = \widetilde{\phi}$ in the cascade algorithm (3.3), and since, according to Theorem 1.2, the support property (1.11) is satisfied, we may appeal to Theorem 3.2(a) to deduce that, for $r = 0, 1, \ldots,$

$$\operatorname{supp}^c \phi_r = \Big[-2n + 1 + \frac{n-\tilde{n}}{2^{r+1}}, 2n - 1 - \frac{n-\tilde{n}}{2^{r+1}} \Big] \subset [-2n+1, 2n-1], \tag{3.71}$$

after having recalled also that $\tilde{n} \leq n$. It follows from (3.68) and (3.71) that $\{\phi_r : r = 0, 1, \ldots\}$ is a Cauchy sequence with respect to the $\|\cdot\|_\infty$ norm in $C_0(\mathbb{R})$, and satisfying

$$\phi_r(x) = 0, \qquad x \notin (-2n+1, 2n-1), \tag{3.72}$$

for each $r = 0, 1, \ldots$.

Hence we may apply a standard result from real analysis (see e.g.[1, p 395,Theorem 13-4]) to deduce the existence of a function $\phi \in C_0(\mathbb{R})$ such that

$$\|\phi - \phi_r\|_\infty \longrightarrow 0, \qquad r \longrightarrow \infty, \tag{3.73}$$

and with, by using also (3.72),

$$\phi(x) = 0, \quad x \notin (-2n+1, 2n-1). \tag{3.74}$$

By using (3.3), together with the fact that, from (3.1), $T_{\mathbf{a}}$ is a linear operator, as well as (3.4), we obtain, for $r = 0, 1, \ldots$,

$$\begin{aligned} \|\phi - T_{\mathbf{a}}\phi\|_\infty &= \|\phi - \phi_{r+1} + T_{\mathbf{a}}\phi_r - T_{\mathbf{a}}\phi\|_\infty \\ &\leq \|\phi - \phi_{r+1}\|_\infty + \left\|T_{\mathbf{a}}(\phi - \phi_r)\right\|_\infty \\ &\leq \|\phi - \phi_{r+1}\|_\infty + \left[\sum_j |a_j|\right] \|\phi - \phi_r\|_\infty \longrightarrow 0, \quad r \longrightarrow \infty, \end{aligned}$$

from (3.73), and thus (3.2) is satisfied, according to which ϕ is a refinable function with refinement sequence $\{a_j\}$.

Finally, since $\widetilde{\phi}$ is assumed to be an interpolatory refinable function, so that $\widetilde{\phi}(j) = \delta_j, j \in \mathbb{Z}$, we may apply Theorem 3.2(b) to deduce that $\phi_r(j) = \delta_j, j \in \mathbb{Z}$, for $r = 0, 1, \ldots$, and thus, by applying also (3.73), it holds for any $j \in \mathbb{Z}$ and $r \in \{0, 1, \ldots\}$ that

$$|\phi(j) - \delta_j| = |\phi(j) - \phi_r(j)| \leq \|\phi - \phi_r\|_\infty \longrightarrow 0, \qquad r \longrightarrow \infty,$$

according to which $\phi(j) = \delta_j$, $j \in \mathbb{Z}$, and it follows that ϕ is indeed an interpolatory refinable function. ■

By virtue of the remark between the statement and proof of Theorem 3.7, the following result is now an immediate consequence of Theorem 3.7.

COROLLARY 3.8 *Let* $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ *denote a sequence for which the corresponding symbol* A*, as defined by (1.21), satisfies* $A \in \mathcal{A}_{m,n}$*, and suppose the subdivision operator* $S_{\boldsymbol{b}_2}$ *defined as in (2.41),(2.38) and (2.35) is contractive, that is,*

$$\|S_{\boldsymbol{b}_1}\|_\infty < 1. \tag{3.75}$$

Then there exists an interpolatory refinable function ϕ *with refinement sequence* $\{a_j\}$*.*

We proceed in Chapter 4 to construct classes of Laurent polynomial mask symbols A in $\mathcal{A}_{m,n}$ such that the conditions of Theorem 3.7 are satisfied.

4 Convergent Interpolatory Subdivision Schemes

In this chapter, we introduce a class of optimally local symmetric interpolatory subdivision schemes satisfying a certain polynomial reproduction property, and proceed to use Theorem 3.7 inductively to prove the existence of the corresponding class of interpolatory refinable functions, and therefore also subdivision convergence as in Theorem 1.6, thereby providing an alternative to other existence and convergence proofs to be found in the literature. As an immediate consequence, we then deduce that the condition (a) of Theorem 3.7 is in fact superfluous.

4.1 Optimally local schemes with prescribed polynomial reproduction

We proceed to present a class of optimally local interpolatory subdivision schemes of the form (1.1) such that the symmetry condition (1.2), as well as, for a prescribed integer $m \in \mathbb{N}$, the π_{2m-1}-polynomial reproduction property (1.3) is satisfied.

To this end we define, as in [5] (see also [18, Equation 5.3]), for any integer $m \in \mathbb{N}$, the Laurent polynomial

$$A_m^I(z) := \frac{1}{2^{2m-1}} z^{-m}(1+z)^{2m} \sum_{j=0}^{m-1} \binom{m+j-1}{j} \left[\tfrac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j, \qquad z \in \mathbb{C} \setminus \{0\}. \quad (4.1)$$

Observe from (4.1) that $A_1^I = A_1$, as given in (1.45).

We see from (4.1) that the sequence $a^{I,m} = \{a_j^{I,m}\} \in M_0(\mathbb{Z})$ defined by

$$\sum_j a_j^{I,m} z^j := A_m^I(z), \qquad z \in \mathbb{C} \setminus \{0\}, \quad (4.2)$$

satisfies

$$\operatorname{supp}\{a_j^{I,m}\} = [-2m+1, 2m-1]|_{\mathbb{Z}}\,, \quad (4.3)$$

and thus

$$A_m^I(z) = \sum_{j=-2m+1}^{2m-1} a_j^{I,m} z^j, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{4.4}$$

According to results in [6, Section 2] and [5, Chapter 8], $\{a_j^{I,m}\}$ is the minimum supported coefficient sequence for which the conditions $(a), \ldots, (d)$ of Theorem 1.1 are satisfied. Note in particular from (4.1) that the symmetry condition (c) of Theorem 1.2 is satisfied by the Laurent polynomial $A = A_m^I$.

Hence, from Theorem 1.2, and (4.1), we have

$$A_m^I \in \mathcal{A}_{m,m}, \tag{4.5}$$

with, moreover,

$$\mathcal{A}_{m,m} = \{A_m^I\}. \tag{4.6}$$

It follows that the Laurent polynomial class $\mathcal{A}_{m,n}$ satisfies $m \leq n$. According to a result in [8, Theorem 1], the Laurent polynomials given by (4.1) provide a basis for the class $\mathcal{A}_{m,n}$, in the following sense.

THEOREM 4.1 *Let the Laurent polynomial* A *satisfy* $A \in \mathcal{A}_{m,n}$. *Then there exists a unique coefficient sequence* $\{t_0, \ldots, t_{n-m}\} \subset \mathbb{R}$, *with*

$$\sum_{j=0}^{n-m} t_j = 1,$$

such that

$$A = \sum_{j=0}^{n-m} t_j A_{m+j}^I,$$

with $\{A_m^I, \ldots, A_n^I\}$ *defined as in (4.1).*

Since $A = A_m^I$ satisfies conditions $(a), \ldots, (d)$ of Theorem 1.2 with $n = m$, we conclude from Theorem 1.2 that the interpolatory subdivision scheme (1.18), with $a = \{a_j\} = \{a_j^{I,m}\}$ is indeed an optimally local interpolatory subdivision scheme such that the symmetry condition (1.12), as well as the π_{2m-1}-polynomial reproduction property (1.13) are satisfied. The optimality locality of this scheme is due to the fact that the support

(4.3) of the sequence $a = \{a_j\} = \{a_j^{I,m}\}$ is minimal. The resulting symmetric interpolatory subdivision scheme

$$c_j^{(0)} := c_j \quad ; \quad c_j^{(r+1)} := (S_m c^{(r)})_j = (S_m^r c)_j, \quad j \in \mathbb{Z}, \tag{4.7}$$

for any control point sequence $c = \{c_j\} \in M(\mathbb{Z})$, where the subdivision operator $S_m : M(\mathbb{Z}) \longrightarrow M(\mathbb{Z})$ is defined, according to (1.17), by

$$(S_m c)_j := \sum_k a_{j-2k}^{I,m} c_k, \qquad j \in \mathbb{Z}, \tag{4.8}$$

is often referred to as the Dubuc-Deslauriers subdivision scheme, due to the introduction thereof in [12] and [11].

According to a result in [6] (see also [8, 7]), we have the explicit formulation

$$a_{2j+1}^{I,m} = \frac{m}{2^{4m-3}} \binom{2m-1}{m} \frac{(-1)^j}{2j+1} \binom{2m-1}{m+j}, \quad j = -m, \ldots, m-1. \tag{4.9}$$

By setting $m = 1, \ldots, 5$ in (4.9), we obtain Table 4.1.

Table 4.1: *The sequences* $\left\{a_{2j+1}^{I,m} : j = -m, \ldots, m-1\right\}, \quad m = 1, \ldots, 5.$

m	$\left\{a_{2j+1}^{I,m} : j = -m, \ldots, m-1\right\}$
1	$\frac{1}{2}\left\{1, 1\right\}$
2	$\frac{1}{16}\left\{-1, 9, 9, -1\right\}$
3	$\frac{1}{256}\left\{3, -25, 150, 150, -25, 3\right\}$
4	$\frac{1}{2048}\left\{-5, 49, -245, 1225, 1225, -245, 49, -5\right\}$
5	$\frac{1}{65536}\left\{35, -405, 2268, -8820, 39690, 39690, -8820, 2268, -405, 35\right\}$

4.2 Existence based on symbol positivity on the unit circle

We present here a set of sufficient conditions on a mask symbol A for the existence of a corresponding interpolatory refinable function ϕ, and thus for the convergence of the associated interpolatory subdivision scheme, as was proved by means of Fourier transform methods in [25, Theorem 4.1 and Corollary 4.1].

The result is as follows.

THEOREM 4.2 *Let* $\{a_j\} \in M_0(\mathbb{Z})$ *be a sequence satisfying the conditions*

$$a_{2j} = \delta_j, \qquad j \in \mathbb{Z}, \tag{4.10}$$

and

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1. \tag{4.11}$$

Suppose that, moreover, the corresponding Laurent polynomial symbol A*, as defined by (1.21), satisfies the positivity condition*

$$A(e^{i\theta}) > 0, \qquad \theta \in (-\pi, \pi). \tag{4.12}$$

Then there exists an interpolatory refinable function ϕ *with refinement sequence* $\{a_j\}$.

Observe that, for a Laurent polynomial $A \in \mathcal{A}_{m,n}$, for which, according to Theorem 1.2, the symmetry condition (1.12) is satisfied, we have from (1.23) and (1.12) that, for any $\theta \in \mathbb{R}$,

$$\begin{aligned}
A(e^{i\theta}) &= \sum_{j=-2n+1}^{2n-1} a_j (e^{i\theta})^j \\
&= \sum_{j=-2n+1}^{2n-1} a_j e^{ji\theta} \\
&= \sum_{j=-2n+1}^{-1} a_j e^{ji\theta} + a_0 + \sum_{j=1}^{2n-1} a_j e^{ji\theta} \\
&= \sum_{j=1}^{2n-1} a_{-j} e^{-ji\theta} + a_0 + \sum_{j=1}^{2n-1} a_j e^{ji\theta} \\
&= a_0 + 2\sum_{j=1}^{2n-1} a_j \left[\frac{e^{ji\theta} + e^{-ji\theta}}{2}\right] \\
&= a_0 + 2\sum_{j=1}^{2n-1} a_j \cos(j\theta),
\end{aligned} \tag{4.13}$$

and thus

$$A(z) \in \mathbb{R}, \qquad |z| = 1. \tag{4.14}$$

If, moreover, we have $n = m$, we see from (4.6) that $A = A^I_m$, for which it follows from (4.1) that, for any $\theta \in \mathbb{R}$,

$$\begin{aligned} A^I_m(e^{i\theta}) &= \frac{1}{2^{2m-1}}\Big[\frac{(1+e^{i\theta})^2}{e^{i\theta}}\Big]^m \sum_{j=0}^{m-1} \binom{m+j-1}{j} \Big[\tfrac{1}{2}\big(1 - \frac{e^{i\theta}+e^{-i\theta}}{2}\big)\Big]^j \\ &= \frac{1}{2^{2m-1}}\Big[2 + 2\frac{e^{i\theta}+e^{-i\theta}}{2}\Big]^m \sum_{j=0}^{m-1} \binom{m+j-1}{j} \Big[\tfrac{1}{2}(1-\cos\theta)\Big]^j \\ &= \frac{1}{2^{m-1}}(1+\cos\theta)^m \sum_{j=0}^{m-1} \binom{m+j-1}{j} \Big(\sin^2\frac{\theta}{2}\Big)^j. \end{aligned} \tag{4.15}$$

According to (4.15), we have

$$A^I_m(e^{i\theta}) > 0, \qquad \theta \in (-\pi, \pi), \tag{4.16}$$

as was proved also in [25, Corollary 4.1] by means of an integral formulation for $A^I_m(e^{i\theta})$. Moreover, since $A^I_m \in \mathcal{A}_{m,m}$, we know that the sequence $\{a^{I,m}_j\} \in M_0(\mathbb{Z})$ defined by (4.2) satisfies

$$a^{I,m}_{2j} = \delta_j, \qquad j \in \mathbb{Z}, \tag{4.17}$$

with also

$$\sum_j a^{I,m}_{2j} = \sum_j a^{I,m}_{2j+1} = 1, \tag{4.18}$$

as noted before in (1.31). Hence we may deduce from Theorem 4.2 the following existence result.

THEOREM 4.3 *Let* $\{a^{I,m}_j\} \in M_0(\mathbb{Z})$ *be defined as in (4.2) and (4.1). Then there exists an interpolatory refinable function* ϕ^I_m *with refinement sequence* $\{a^{I,m}_j\}$.

As an immediate consequence of Theorems 4.3, 1.4, 1.5, 1.7 and Corollary 1.8, we now have the following result.

THEOREM 4.4 *The interpolatory refinable function* ϕ^I_m *of Theorem 4.3 satisfies the following properties:*

(a) ϕ_m^I *is the only interpolatory refinable function with refinement sequence* $\{a_j^{I,m}\}$.

(b)

$$\phi_m^I\left(\frac{j}{2}\right) = a_j^{I,m}, \qquad j \in \mathbb{Z}. \tag{4.19}$$

(c)

$$\text{supp}^c \phi_m^I = [-2m+1, 2m-1]. \tag{4.20}$$

(d)

$$\phi_m^I(-x) = \phi_m^I(x), \qquad x \in \mathbb{R}. \tag{4.21}$$

(e)

$$\sum_j p(j)\phi_m^I(x-j) = p(x), \qquad x \in \mathbb{R}, \quad p \in \pi_{2m-1}. \tag{4.22}$$

(f)

$$\sum_j \phi_m^I(x-j) = 1, \qquad x \in \mathbb{R}. \tag{4.23}$$

In addition, by appealing also to Theorem 1.6, we have the following subdivision convergence result.

Corollary 4.5 *For the interpolatory subdivision scheme (4.7), with the interpolatory subdivision operator* S_m *given by (4.8), it holds that*

$$\boldsymbol{\Phi}_{m,\boldsymbol{c}}^I\left(\frac{j}{2^r}\right) = \boldsymbol{c}_j^{(r)}, \qquad j \in \mathbb{Z}, \quad r = 0, 1, \ldots, \tag{4.24}$$

where the continuous function $\boldsymbol{\Phi}_{m,\boldsymbol{c}}^I : \mathbb{R} \longrightarrow \mathbb{R}^s$ *is defined by*

$$\boldsymbol{\Phi}_{m,\boldsymbol{c}}^I(x) := \sum_j \boldsymbol{c}_j \phi_m^I(x-j), \qquad x \in \mathbb{R}, \tag{4.25}$$

for any control point sequence $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in M^s(\mathbb{Z})$*. Moreover,* $\boldsymbol{\Phi}_{m,\boldsymbol{c}}^I$ *is the limit curve of (4.7), in the sense that, for any* $x \in \mathbb{R}$*, and a sequence* $\{j_r : r = 0, 1, \ldots\}$ *such that*

$$\left|x - \frac{j_r}{2^r}\right| \longrightarrow 0, \qquad r \longrightarrow \infty, \tag{4.26}$$

the convergence result

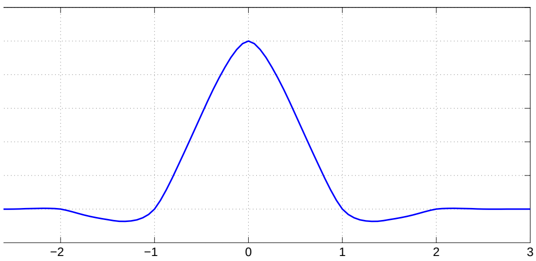
$$\left|\boldsymbol{\Phi}^{I}_{m,c}(x) - \boldsymbol{c}^{(r)}_{j_r}\right| \longrightarrow 0, \qquad r \longrightarrow \infty, \tag{4.27}$$

is satisfied.

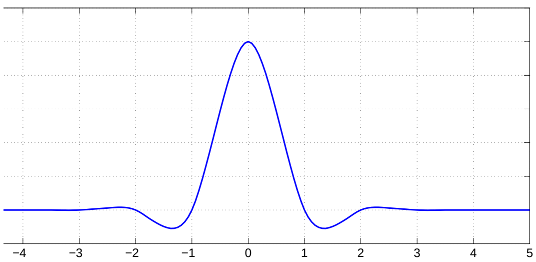
Observe from (4.25) that the choice $\mathbf{c} = \boldsymbol{\delta} = \{\delta_j\}$, the delta sequence, yields $\Phi^{I}_{m,\boldsymbol{\delta}} = \phi^{I}_{m}$, and thus, from (4.24),

$$\phi^{I}_{m}\left(\frac{j}{2^r}\right) = \mathbf{c}^{(r)}_{j}, \qquad j \in \mathbb{Z}, \quad r = 0, 1, \ldots, \tag{4.28}$$

that is, by plotting $\mathbf{c}^{r}_{j}$, $j \in \mathbb{Z}$, (on the y-axis) against $\frac{j}{2^r}$, $j \in \mathbb{Z}$, (on the x-axis) we can generate, for successively increasing values of $r = 0, 1, \ldots$, the graph of ϕ^{I}_{m}. By using (4.7) and Table 4.1, as well as in the Algorithm 4.3.1 in [5], we display in Figures 4.1(a), and 4.1(b), the resulting graphs of ϕ^{I}_{2} and ϕ^{I}_{3} .



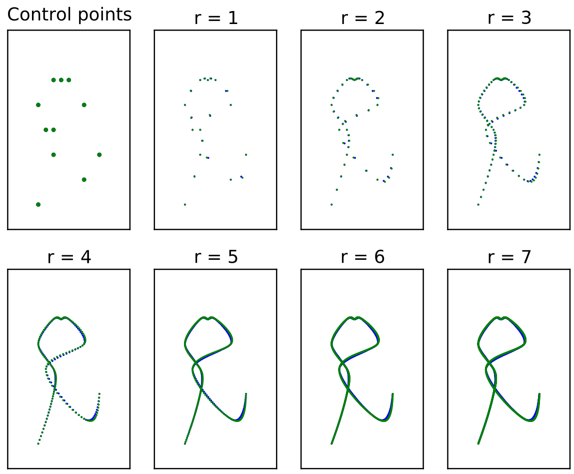
(a) ϕ^{I}_{2}



(b) ϕ^{I}_{3}

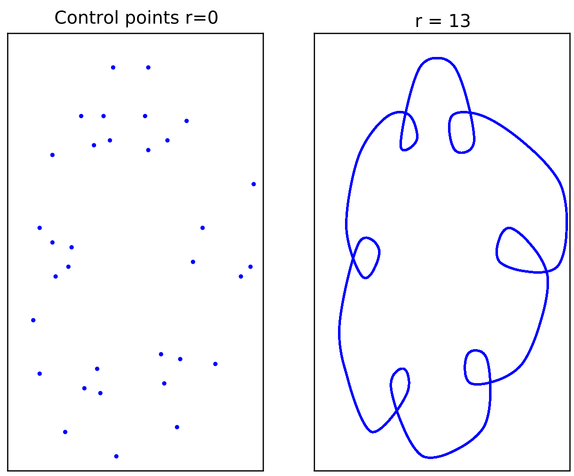
Figure 4.1: *The refinable functions* ϕ^{I}_{2}, *and* ϕ^{I}_{3}.

In Figures 4.2 and 4.3, we show, as a graphical illustration of Corollary 4.5, the generation of subdivision curves for the indicated choice of control points $\mathbf{c} = \{\mathbf{c}_j\}$, with $\mathbf{c}_j \in \mathbb{R}^2$. To accommodate endpoints, we have used Algorithm 8.3.2 in [5].



$\mathbf{c}^{(0)} = \mathbf{c}$ the control points denoted by $(\cdot)$ and $\mathbf{c}^{(7)} \approx \Phi^{I}_{3,\mathbf{c}}\left(\frac{\cdot}{2^7}\right)$ the limit curve after 7 iterations denoted by $(-)$.

Figure 4.2: *Illustration of the interpolatory subdivision scheme (4.7) with* $m = 3$.



$\mathbf{c}^{(0)} = \mathbf{c}$ the control points denoted by $(\cdot)$ and $\mathbf{c}^{(13)} \approx \Phi^{I}_{2,\mathbf{c}}\left(\frac{\cdot}{2^{13}}\right)$ the limit curve after 13 iterations denoted by $(-)$.

Figure 4.3: *Illustration of the interpolatory subdivision scheme (4.7) with* $m = 2$.

As shown in [5] (see also [28, Theorem 4.11], we have $\phi^I_2 \in C^1_0(\mathbb{R})$ and $\phi^I_3 \in C^2_0(\mathbb{R})$.

4.3 Existence based on contractivity

In this section, we proceed to show how Theorem 3.7 can be used recursively to obtain, as an alternative to the proof given in [25, Theorem 4.1], an interpolatory refinable function existence proof for ϕ_m^I, as based on cascade algorithm convergence. Also, since we may then always choose $\widetilde{\phi} = \phi_\ell^I$ for any $\ell \in \{1, \ldots, m\}$ in Theorem 3.7, it will follow that condition (a) of Theorem 3.7 becomes superfluous, on the basis of which the contractive condition (3.40) is then an alternative to the positivity condition (4.12) of Theorem 4.2 as a sufficient condition for the existence of an interpolatory refinable function, and thereby making the interpolatory refinable function theory presented in this thesis self-contained.

In the following we therefore repeat the formulation of Theorem 4.3, and give a proof thereof which is based on a recursive application of Theorem 3.7.

THEOREM 4.6 *Let* $\{a_j^{I,m}\} \in M_0(\mathbb{Z})$ *be defined as in (4.2) and (4.1). Then there exists an interpolatory refinable function* ϕ_m^I *with refinement sequence* $\{a_j^{I,m}\}$*.*

Proof. Our proof is by mathematical induction. First, observe that, since, from (4.9) and (4.17), we have

$$\left\{a_{-1}^{I,1}, a_0^{I,1}, a_1^{I,1}\right\} = \left\{\tfrac{1}{2}, 1, \tfrac{1}{2}\right\} \qquad , \qquad a_j^{I,1} = 0, \quad |j| > 1, \tag{4.29}$$

which corresponds precisely with (1.19), we deduce that the result of the theorem holds for $m = 1$, with $\phi_1^I = h$, the hat function as given by (1.51).

Next, for a fixed integer $m \in \mathbb{N}$, suppose $\{a_j^{I,m}\} \in M_0(\mathbb{Z})$ is given by (4.1) and (4.2), and there exists an interpolatory function ϕ_m^I with refinement sequence $\{a_j^{I,m}\}$. Now let $\{a_j^{I,m+1}\} \in M_0(\mathbb{Z})$ be defined by replacing m by $m+1$ in (4.1) and (4.2). Our inductive proof will be complete if we can show that there exists an interpolatory refinable function ϕ_{m+1}^I with refinement sequence $\{a_j^{I,m+1}\}$.

To this end, we first recall from (4.5) that the Laurent polynomial symbol A_{m+1}^I satisfies

$$A^{I}_{m+1} \in \mathcal{A}_{m+1,m+1}. \tag{4.30}$$

Hence, if we choose $\{a_j\} = \{a^{I,m+1}_j\}$ in Theorem 3.7, then condition (a) of that theorem is satisfied with $\ell = m$, $\tilde{n} = m$, $\widetilde{\phi} = \phi^{I}_{m}$ and $\{\tilde{a}_j\} = \{a^{I,m}_j\}$.

In order to formulate condition (b) of Theorem 3.7 for the choice $\{a_j\} = \{a^{I,m+1}_j\}$, we note from (4.1) that

$$A^{I}_{m+1}(z) = (1+z)^{2m} B^{I,m+1}_{m}(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{4.31}$$

where

$$B^{I,m+1}_{m}(z) := \frac{1}{2^{2m+1}} z^{-m-1}(1+z)^2 \sum_{j=0}^{m} \binom{m+j}{j} \left[\tfrac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{4.32}$$

Let the sequence $\mathbf{b}^{I,m+1} = \{b^{I,m+1}_j\} \in M_0(\mathbb{Z})$ be defined by

$$\sum_j b^{I,m+1}_j z^j := B^{I,m+1}_{m}(z), \qquad z \in \mathbb{C} \setminus \{0\}. \tag{4.33}$$

It remains to prove that the subdivision operator $S_{\mathbf{b}^{I,m+1}}$ is contractive, that is,

$$\|S_{\mathbf{b}^{I,m+1}}\|_\infty < 1, \tag{4.34}$$

for then condition (b) of Theorem 3.7 is satisfied for $\ell = m$, so that, according to Theorem 3.7, there exists an interpolatory refinable function ϕ^{I}_{m+1} with refinement sequence $\{a^{I,m+1}_j\}$, and thereby completing our inductive proof.

We shall in fact prove that

$$\rho_{m+1} := \max\left\{ \sum_j \left|b^{I,m+1}_{2j}\right| \quad , \quad \sum_j \left|b^{I,m+1}_{2j+1}\right| \right\} < 1, \tag{4.35}$$

which, according to (2.5) in Theorem 2.1, is equivalent to (4.34). To this end, we first define the polynomial P by means of

$$P(z) := z^m \sum_{j=0}^{m} \binom{m+j}{j} \left[\tfrac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{4.36}$$

according to which $\deg(P) = 2m$, and let the sequence $\{p_j\} \in M_0(\mathbb{Z})$ be defined by

$$\sum_j p_j z^j = \sum_{j=0}^{2m} p_j z^j := P(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{4.37}$$

so that, from (4.37) and (4.36),

$$\operatorname{supp}\{p_j\} = [0, 2m]|_{\mathbb{Z}}. \tag{4.38}$$

Observe from (4.32) and (4.36) that

$$B_m^{I,m+1}(z) = \frac{1}{2^{2m+1}} z^{-2m-1}(1+z)^2 P(z), \qquad z \in \mathbb{C} \setminus \{0\}. \tag{4.39}$$

Now note that, for any $j \in \{0, \ldots, m\}$ and $z \in \mathbb{C} \setminus \{0\}$, we have

$$\left[\tfrac{1}{2}\left(1 - \frac{z+z^{-1}}{2}\right)\right]^j = \frac{(-1)^j}{2^{2j}} \frac{(z-1)^{2j}}{z^j},$$

and thus, from (4.36), by recalling also the convention $\binom{\ell}{k} := 0$ if $k < 0$ or $k > \ell$, we obtain

$$\begin{aligned}
P(z) &= \sum_{j=0}^{m} (-1)^j \frac{1}{2^{2j}} \binom{m+j}{j} (z-1)^{2j} z^{m-j} \\
&= \sum_{j=0}^{m} (-1)^j \frac{1}{2^{2j}} \binom{m+j}{j} \sum_k (-1)^k \binom{2j}{k} z^{k+m-j} \\
&= \sum_{j=0}^{m} (-1)^j \frac{1}{2^{2j}} \binom{m+j}{j} (-1)^{m-j} \sum_k (-1)^k \binom{2j}{k-m+j} z^k \\
&= (-1)^m \sum_{k=0}^{2m} (-1)^k \Big[\sum_{j=m-k}^{m} \frac{1}{2^{2j}} \binom{m+j}{j} \binom{2j}{k-m+j} \Big] z^k,
\end{aligned}$$

which, together with (4.37), yields

$$p_j = (-1)^{m+j} \sum_{k=m-j}^{m} \frac{1}{2^{2k}} \binom{m+k}{k} \binom{2k}{j-m+k}, \quad j = 0, 1, \ldots, 2m,$$

or equivalently,

$$p_j = (-1)^{m+j} \sum_{k=m-j}^{m} \frac{1}{2^{2k}} \binom{m+k}{k} \binom{2k}{k+m-j}, \quad j = 0, 1, \ldots, 2m. \tag{4.40}$$

Note in particular from (4.40) that

$$\left.\begin{aligned}
(-1)^m p_{2j} &> 0 \qquad j = 0, \ldots, m; \\
(-1)^m p_{2j+1} &< 0 \qquad j = 0, \ldots, m-1.
\end{aligned}\right\} \tag{4.41}$$

It follows from (4.41) and (4.38) that

$$\left.\begin{aligned} \sum_j |p_{2j}| &= (-1)^m \sum_j p_{2j}; \\ \sum_j |p_{2j+1}| &= (-1)^{m+1} \sum_j p_{2j+1}. \end{aligned}\right\} \tag{4.42}$$

Next, we observe from (4.36) that

$$P(1) = 1, \tag{4.43}$$

whereas

$$\begin{aligned} P(-1) &= (-1)^m \sum_{j=0}^{m} \binom{m+j}{j} \\ &= (-1)^m \sum_{j=0}^{m} \left[\binom{m+j+1}{j} - \binom{m+j}{j-1}\right] \\ &= (-1)^m \left\{\left[\binom{m+1}{0} - 0\right] + \left[\binom{m+2}{1} - \binom{m+1}{0}\right]\right. \\ &+ \left.\left[\binom{m+3}{2} - \binom{m+2}{1}\right] + \cdots + \left[\binom{2m+1}{m} - \binom{2m}{m-1}\right]\right\} \\ &= (-1)^m \binom{2m+1}{m}. \end{aligned} \tag{4.44}$$

It follows from (4.37), (4.43) and (4.44) that

$$\left.\begin{aligned} 1 &= \sum_j p_j = \sum_j p_{2j} + \sum_j p_{2j+1}; \\ (-1)^m \binom{2m+1}{m} &= \sum_j (-1)^j p_j = \sum_j p_{2j} - \sum_j p_{2j+1}, \end{aligned}\right\}$$

which yields the formulas

$$\left.\begin{aligned} \sum_j p_{2j} &= \tfrac{1}{2}\left[1 + (-1)^m \binom{2m+1}{m}\right]; \\ \sum_j p_{2j+1} &= \tfrac{1}{2}\left[1 + (-1)^{m+1} \binom{2m+1}{m}\right]. \end{aligned}\right\} \tag{4.45}$$

It follows from (4.42) and (4.45) that

$$\left.\begin{aligned}\sum_j |p_{2j}| &= \tfrac{1}{2}\Big[\binom{2m+1}{m} + (-1)^m\Big];\\ \sum_j |p_{2j+1}| &= \tfrac{1}{2}\Big[\binom{2m+1}{m} + (-1)^{m+1}\Big].\end{aligned}\right\} \tag{4.46}$$

Now use (4.33) and (4.39) to deduce that, for any $z \in \mathbb{C}\setminus\{0\}$,

$$\begin{aligned}\sum_j b_j^{I,m+1} z^j &= \frac{1}{2^{2m+1}} z^{-2m-1}(1+2z+z^2)\sum_j p_j z^j\\ &= \frac{1}{2^{2m+1}}\Big[\sum_j p_j z^{j-2m-1} + 2\sum_j p_j z^{j-2m} + \sum_j p_j z^{j-2m+1}\Big]\\ &= \frac{1}{2^{2m+1}}\Big[\sum_j p_{j+2m+1} z^j + 2\sum_j p_{j+2m} z^j + \sum_j p_{j+2m-1} z^j\Big]\\ &= \frac{1}{2^{2m+1}}\sum_j \Big(p_{j+2m+1} + 2p_{j+2m} + p_{j+2m-1}\Big) z^j,\end{aligned}$$

and thus

$$b_j^{I,m+1} = \frac{1}{2^{2m+1}}\Big(p_{j+2m+1} + 2p_{j+2m} + p_{j+2m-1}\Big), \quad j \in \mathbb{Z}. \tag{4.47}$$

Hence, from (4.47) and (4.46), we have

$$\begin{aligned}\sum_j \left|b_{2j}^{I,m+1}\right| &\leq \frac{1}{2^{2m+1}}\Big[\sum_j |p_{2j+2m+1}| + 2\sum_j |p_{2j+2m}| + \sum_j |p_{2j+2m-1}|\Big]\\ &= \frac{1}{2^{2m}}\Big[\sum_j |p_{2j+1}| + \sum_j |p_{2j}|\Big]\\ &= \frac{1}{2^{2m+1}}\Big[\binom{2m+1}{m} + (-1)^{m+1} + \binom{2m+1}{m} + (-1)^m\Big]\\ &= \frac{1}{2^{2m}}\binom{2m+1}{m},\end{aligned} \tag{4.48}$$

and similarly,

$$
\begin{aligned}
\sum_j \left|b_{2j+1}^{I,m+1}\right| &\leq \frac{1}{2^{2m+1}}\Big[\sum_j |p_{2j+2m+2}| + 2\sum_j |p_{2j+2m+1}| + \sum_j |p_{2j+2m}|\Big] \\
&= \frac{1}{2^{2m}}\Big[\sum_j |p_{2j+1}| + \sum_j |p_{2j}|\Big] \\
&= \frac{1}{2^{2m+1}}\Big[\binom{2m+1}{m} + (-1)^m + \binom{2m+1}{m} + (-1)^{m+1}\Big] \\
&= \frac{1}{2^{2m}}\binom{2m+1}{m}. \qquad (4.49)
\end{aligned}
$$

It follows from the definition of ρ_{m+1} in (4.35), together with (4.48) and (4.49), that

$$
\rho_{m+1} \leq \frac{1}{2^{2m}}\binom{2m+1}{m}. \qquad (4.50)
$$

Hence the desired result (4.35) will follow if we can show that

$$
\binom{2m+1}{m} < 2^{2m}, \qquad m \in \mathbb{N}, \qquad (4.51)
$$

which we proceed to prove inductively. Since $\binom{3}{1} = 3 < 4 = 2^2$, we see that (4.51) holds for $m = 1$. Suppose next (4.51) holds for a fixed integer $m \in \mathbb{N}$. But then

$$
\begin{aligned}
\binom{2m+3}{m+1} &= \binom{2m+2}{m+1} + \binom{2m+2}{m} \\
&= \Big[\binom{2m+1}{m+1} + \binom{2m+1}{m}\Big] + \Big[\binom{2m+1}{m} + \binom{2m+1}{m-1}\Big] \\
&= \binom{2m+1}{m+1} + 2\binom{2m+1}{m} + \binom{2m+1}{m-1} \\
&= 3\binom{2m+1}{m} + \frac{(2m+1)!}{(m-1)!(m+2)!} \\
&= 3\binom{2m+1}{m} + \frac{m}{m+2}\frac{(2m+1)!}{(m+1)!m!} \\
&= \Big(3 + \frac{m}{m+2}\Big)\binom{2m+1}{m} \\
&< \Big(3 + \frac{m}{m+2}\Big)2^{2m} \\
&< (3+1)2^{2m} = 2^{2m+2},
\end{aligned}
$$

according to which (4.51) holds with m replaced by $m + 1$, and thereby concluding our inductive proof of (4.51). ■

Now observe that, by virtue of Theorem 4.6, condition (a) of Theorem 3.7 is in fact superfluous. Indeed, if $a = \{a_j\} \in M_0(\mathbb{Z})$ is a sequence for which the corresponding symbol A satisfies $A \in \mathcal{A}_{m,n}$, and the contractivity condition (3.40) holds for some integer $\ell \in \{1, \dots, m\}$, then condition (a) of Theorem 3.7 is satisfied by $\widetilde{\phi} = \phi_\ell^I$, $\{\tilde{a}_j\} = \{a_j^{I,\ell}\}$, and $\widetilde{A} = A_\ell^I \in \mathcal{A}_{\ell,\ell}$, so that $\tilde{n} = \ell \leq m \leq n$.

By recalling also Theorems 1.6 and 1.7, we have the following existence and convergence result for interpolatory refinable functions and subdivision, which improves on the general subdivision convergence result of Theorem 2.6, and which can be used as an alternative to Theorem 4.2.

THEOREM 4.7 *Let the sequence $\boldsymbol{a} = \{a_j\} \in M_0(\mathbb{Z})$ be such that its corresponding Laurent polynomial A, as given by (1.21), belongs to the class $\mathcal{A}_{m,n}$, and suppose there is an integer $\ell \in \{1, \dots, m\}$ for which the subdivision operator $S_{\boldsymbol{b}_\ell}$, as defined by (2.41), (2.37) and (2.35), is contractive, that is,*

$$\|S_{\boldsymbol{b}_\ell}\|_\infty < 1. \tag{4.52}$$

Then there exists an interpolatory refinable function ϕ with refinement sequence $\{a_j\}$. Moreover, ϕ satisfies the properties (a),(b) and (c) of Theorem 1.7, and the interpolatory subdivision scheme (1.18) is convergent for any control point sequence $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in M^s(\mathbb{Z})$, in the sense of Theorem 1.6.

5 A one-parameter Class of Interpolatory Subdivision Schemes.

In this chapter, we apply the existence and convergence results of Chapter 4 to the one-parameter family of mask sequences corresponding to the class $\mathcal{A}_{m,m+1}$ of Laurent polynomial mask symbols. In particular, we show that, for $m = 1, 2, 3$, our Theorem 4.7 yields a parameter intervals for existence and convergence which improve on the ones previously obtained in [8, Proposition 3] from an application of Theorem 4.2.

5.1 Formulation and convergence based on Theorem 4.2.

Motivated by Theorem 4.1, we define, for any $m \in \mathbb{N}$, the one-parameter class of Laurent polynomial symbols

$$A_m^I(t|z) := (1-t)A_m^I(z) + tA_{m+1}^I(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{5.1}$$

with $t \in \mathbb{R}$ denoting a parameter, and with A_m^I as defined in (4.1). According to Theorem 4.1, we then have

$$\left\{A_m^I(t|\cdot) : t \in \mathbb{R} \setminus \{0\}\right\} = \mathcal{A}_{m,m+1}. \tag{5.2}$$

We shall denote by $\mathbf{a}^{I,m}(t) = \{a_j^{I,m}(t)\}$ the sequence in $M_0(\mathbb{Z})$ defined for any $t \in \mathbb{R}$ by

$$\sum_j a_j^{I,m}(t) z^j := A_m^I(t|z), \qquad z \in \mathbb{C} \setminus \{0\}. \tag{5.3}$$

The parameter t can be interpreted as a shape parameter in the context of interpolatory subdivision. Note from (5.1) that

$$\left.\begin{array}{rcl} A_m^I(0|z) &=& A_m^I(z), \\ A_m^I(1|z) &=& A_{m+1}^I(z), \end{array}\right\}, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{5.4}$$

It furthermore follows from (5.1) and (4.1) that, for any $t \in \mathbb{R}$,

$$A_m^I(t|z) = (1+z)^{2m} B_m^I(t|z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{5.5}$$

where

$$\begin{aligned} B_m^I(t|z) \; &:= \; \frac{1}{2^{2m+1}} z^{-m-1} \Big\{ 4(1-t)z \sum_{j=0}^{m-1} \binom{m+j-1}{j} \Big[\tfrac{1}{2}(1 - \frac{z+z^{-1}}{2}) \Big]^j \\ &+ \; t(1+z)^2 \sum_{j=0}^{m} \binom{m+j}{j} \Big[\tfrac{1}{2}(1 - \frac{z+z^{-1}}{2}) \Big]^j \Big\}, \qquad z \in \mathbb{C} \setminus \{0\}. \end{aligned} \tag{5.6}$$

We proceed to apply Theorem 4.2, to obtain a $t-$interval for which there exists an interpolatory refinable function $\phi_m^I(t|\cdot)$ with refinement sequence $\{a_j^{I,m}(t)\}$, that is,

$$\phi_m^I(t|x) = \sum_j a_j^{I,m}(t) \phi_m^I(t|2x - j), \qquad x \in \mathbb{R}, \tag{5.7}$$

and for which the interpolatory subdivision scheme

$$c_j^{(0)} = c_j \quad ; \quad c_j^{(r+1)} = \sum_k a_{j-2k}^{I,m}(t) c_k^{(r)}, \qquad j \in \mathbb{Z}, \tag{5.8}$$

therefore converges in the sense of Theorem 1.6, and for any control point sequence $\mathbf{c} = \{\mathbf{c}_j\} \in M^s(\mathbb{Z})$, to the limit function

$$\mathbf{\Phi}_{m,\mathbf{c}}^I(t|x) := \sum_j \mathbf{c}_j \phi_m^I(t|x - j), \qquad x \in \mathbb{R}. \tag{5.9}$$

Our application of Theorem 4.2 is based on the following result from [8, Proposition 3]

THEOREM 5.1 *For $m \in \mathbb{N}$ and $t \in \mathbb{R}$, let the sequence $\mathbf{a}^{I,m}(t) = \{a_j^{I,m}(t)\}$ be defined by (5.3), (5.1) and (4.1). Then the positivity condition (4.12) of Theorem 4.2 is satisfied for*

$$t \in \Big[-2m, 1 \Big]. \tag{5.10}$$

Moreover, if $t > 1$, then the Laurent polynomial symbol $A_m^I(t|\cdot)$ has precisely two zeros on the unit circle in $\mathbb{C}$, and the positivity condition (4.12) is therefore violated.

It follows from Theorems 5.1 and 4.2 that interpolatory refinable function existence and subdivision convergence with respect to the sequence $\{a_j^{I,m}(t)\}$, for respectively $m = 1, 2$, or 3, are guaranteed for the t−intervals

$$\left.\begin{array}{ll} m = 1: & t \in \big[-2, 1\big], \\ \\ m = 2: & t \in \big[-4, 1\big], \\ \\ m = 3: & t \in \big[-6, 1\big]. \end{array}\right\} \tag{5.11}$$

Moreover, if $t > 1$, it follows from the last statement in Theorem 5.1 that the positivity condition (4.12) of Theorem 4.2 is not satisfied, according to which Theorem 4.2 is inconclusive about interpolatory refinable function existence and subdivision convergence for $t > 1$.

By setting $m = 1, 2, 3$ in (5.1), and using (4.4) and Table 4.1, we obtain, for $t \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \{0\}$, the Laurent polynomials

$$\begin{aligned} A_1^I(t|z) &= (1-t)A_1^I(z) + tA_2^I(z) \\ &= a_{-3}^{I,1}(t)z^{-3} + a_{-1}^{I,1}(t)z^{-1} + 1 + a_1^{I,1}(t)z + a_3^{I,1}(t)z^3 \\ &= -\frac{t}{16}z^{-3} + \Big(\frac{1}{2} + \frac{t}{16}\Big)z^{-1} + 1 + \Big(\frac{1}{2} + \frac{t}{16}\Big)z - \frac{t}{16}z^3, \end{aligned} \tag{5.12}$$

$$\begin{aligned} A_2^I(t|z) &= (1-t)A_2^I(z) + tA_3^I(z) \\ &= a_{-5}^{I,2}(t)z^{-5} + a_{-3}^{I,2}(t)z^{-3} + a_{-1}^{I,2}(t)z^{-1} + 1 + a_1^{I,2}(t)z + a_3^{I,2}(t)z^3 + a_5^{I,2}(t)z^5 \\ &= \frac{3t}{2^8}z^{-5} - \Big(\frac{1}{16} + \frac{9t}{2^8}\Big)z^{-3} + \Big(\frac{9}{16} + \frac{6t}{2^8}\Big)z^{-1} + 1 \\ &\quad + \Big(\frac{9}{16} + \frac{6t}{2^8}\Big)z - \Big(\frac{1}{16} + \frac{9t}{2^8}\Big)z^3 + \frac{3t}{2^8}z^5, \end{aligned} \tag{5.13}$$

and

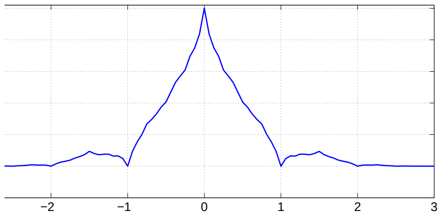
$$
\begin{aligned}
A_3^I(t|z) &= (1-t)A_3^I(z) + tA_4^I(z) \\
&= a_{-7}^{I,3}(t)z^{-7} + a_{-5}^{I,2}(t)z^{-5} + a_{-3}^{I,2}(t)z^{-3} + a_{-1}^{I,2}(t)z^{-1} + 1 + a_1^{I,2}(t)z + a_3^{I,2}(t)z^3 \\
&\quad + a_5^{I,2}(t)z^5 + a_7^{I,3}(t)z^7 \\
&= -\frac{5t}{2^{11}}z^{-7} + \Big(\frac{3}{2^8} + \frac{25t}{2^{11}}\Big)z^{-5} - \Big(\frac{25}{2^8} + \frac{45t}{2^{11}}\Big)z^{-3} \\
&\quad + \Big(\frac{150}{2^8} + \frac{25t}{2^{11}}\Big)z^{-1} + 1 + \Big(\frac{150}{2^8} + \frac{25t}{2^{11}}\Big)z \\
&\quad - \Big(\frac{25}{2^8} + \frac{45t}{2^{11}}\Big)z^3 + \Big(\frac{3}{2^8} + \frac{25t}{2^{11}}\Big)z^5 - \frac{5t}{2^{11}}z^7. \qquad (5.14)
\end{aligned}
$$

Hence, calculating by means of (5.12), (5.13) and (5.14), we obtain Table 5.1 below.

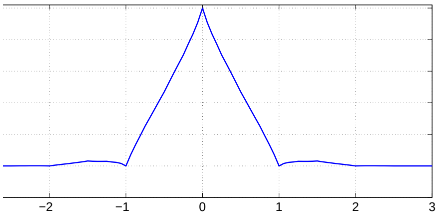
Table 5.1: *The sequences* $\{a_j^{I,m}(t)\}$ *on their supports for* $m = 1, 2, 3$ *and* t *inside the convergence intervals (5.11).*

m	t	$\Big\{a_j^{I,m}(t)\Big\}_{j=-2m-1}^{j=2m+1}$
1	$-\frac{3}{2}$	$\frac{1}{32}\Big\{3, 0, 13, 32, 13, 0, 3\Big\}$
1	$-\frac{1}{2}$	$\frac{1}{32}\Big\{1, 0, 15, 32, 15, 0, 1\Big\}$
1	$\frac{1}{2}$	$\frac{1}{32}\Big\{-1, 0, 17, 32, 17, 0, -1\Big\}$
2	$-\frac{7}{2}$	$\frac{1}{512}\Big\{-21, 0, 31, 0, 246, 512, 246, 0, 31, 0, -21\Big\}$
2	$-\frac{1}{2}$	$\frac{1}{512}\Big\{-3, 0, -23, 0, 282, 512, 282, 0, -23, 0, -3\Big\}$
2	$\frac{1}{2}$	$\frac{1}{512}\Big\{3, 0, -41, 0, 294, 512, 294, 0, -41, 0, 3\Big\}$
3	$-\frac{11}{2}$	$\frac{1}{4096}\Big\{55, 0, -227, 0, 95, 0, 2125, 4096, 2125, 0, 95, 0, -227, 0, 55\Big\}$
3	$-\frac{1}{2}$	$\frac{1}{4096}\Big\{5, 0, 23, 0, -355, 0, 2375, 4096, 2375, 0, -355, 0, 23, 0, 5\Big\}$
3	$\frac{1}{2}$	$\frac{1}{4096}\Big\{-5, 0, 73, 0, -445, 0, 2425, 4096, 2425, 0, -445, 0, 73, 0, -5\Big\}$

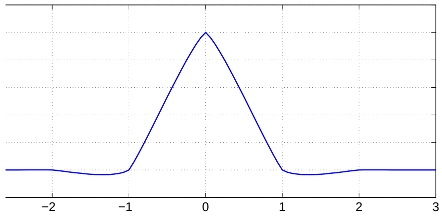
Graphical illustrations are provided in Figures 5.1 − 5.3, in which we applied Algorithm 4.3.1 in [5] to obtain the graphs of the interpolatory refinable function $\phi_m(t|\cdot)$, whereas Algorithm 8.3.2 in [5] was applied to generate, for control point sequences $\mathbf{c} = \{\mathbf{c}_j\}$ as shown, the corresponding limit subdivision curves $\mathbf{\Phi}_{m,\mathbf{c}}^{I}(t|\cdot)$



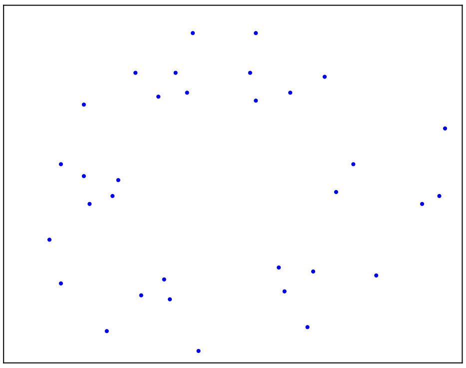
(a) $\phi_1^{\mathrm{I}}(-\frac{3}{2}|\cdot)$



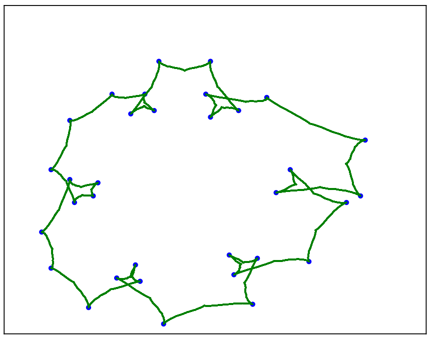
(b) $\phi_1^{\mathrm{I}}(-\frac{1}{2}|\cdot)$



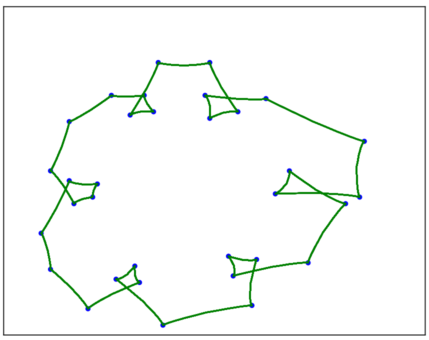
(c) $\phi_1^{\mathrm{I}}(\frac{1}{2}|\cdot)$



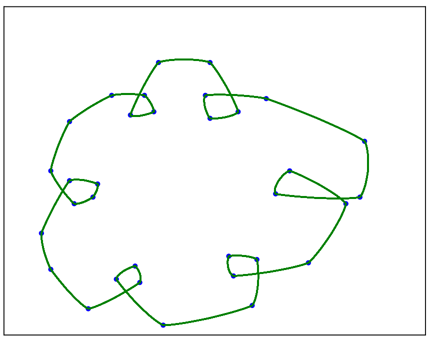
(d) Control points



(e) $\mathbf{\Phi}_1^{\mathrm{I}}(-\frac{3}{2}|\cdot)$

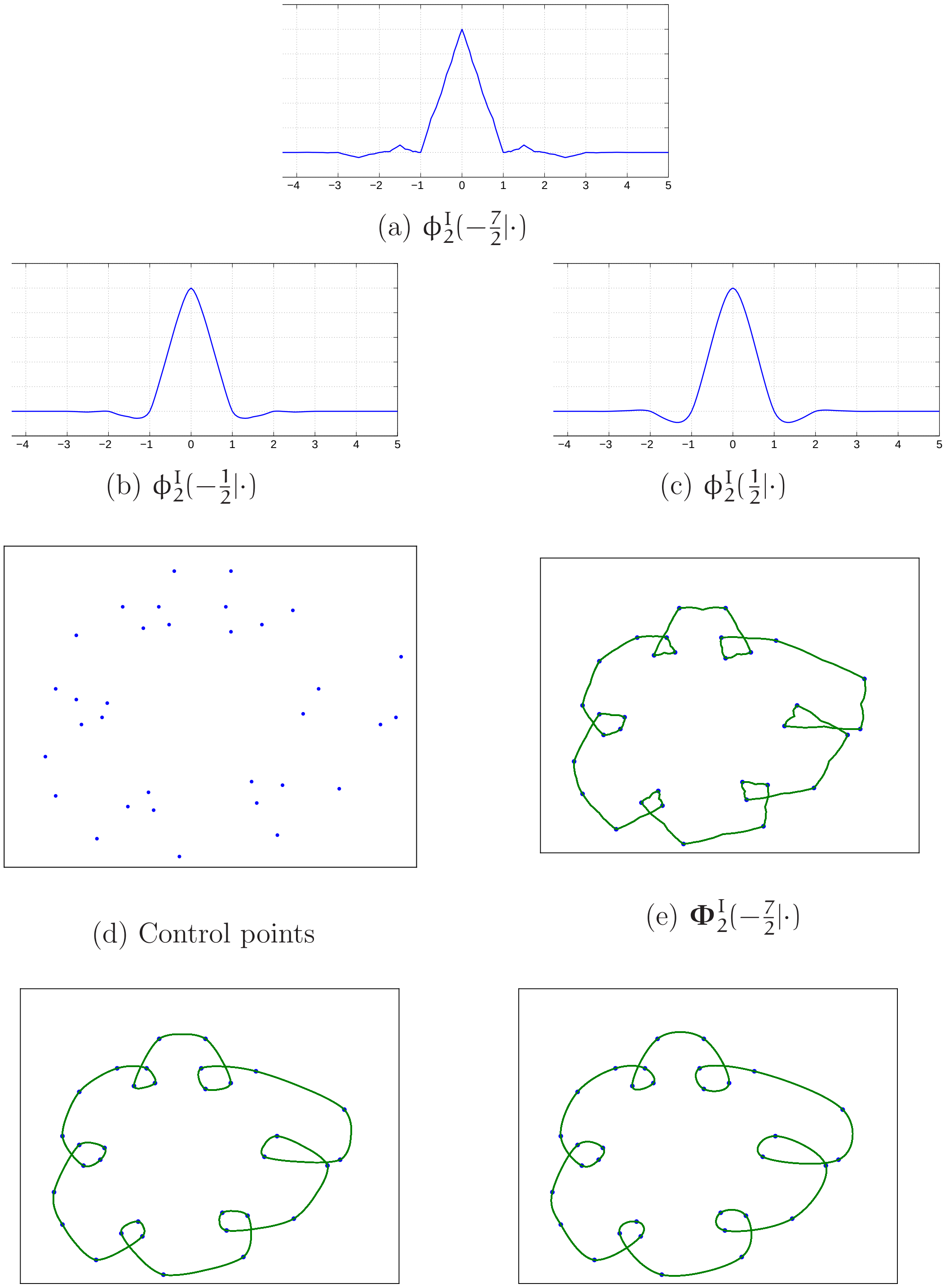


(f) $\mathbf{\Phi}_1^{\mathrm{I}}(-\frac{1}{2}|\cdot)$



(g) $\mathbf{\Phi}_1^{\mathrm{I}}(\frac{1}{2}|\cdot)$

Figure 5.1: *The case* $m = 1$ *of Table 5.1.*



(a) $\phi_2^{\mathrm{I}}(-\frac{7}{2}|\cdot)$

(b) $\phi_2^{\mathrm{I}}(-\frac{1}{2}|\cdot)$

(c) $\phi_2^{\mathrm{I}}(\frac{1}{2}|\cdot)$

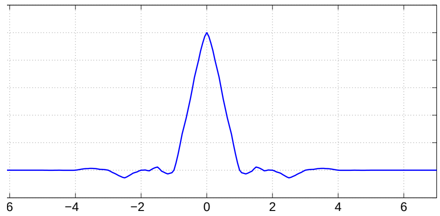
(d) Control points

(e) $\boldsymbol{\Phi}_2^{\mathrm{I}}(-\frac{7}{2}|\cdot)$

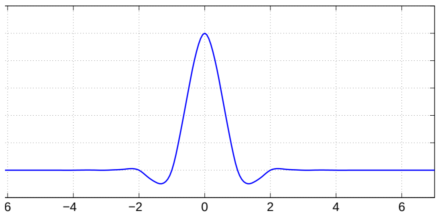
(f) $\boldsymbol{\Phi}_2^{\mathrm{I}}(-\frac{1}{2}|\cdot)$

(g) $\boldsymbol{\Phi}_2^{\mathrm{I}}(\frac{1}{2}|\cdot)$

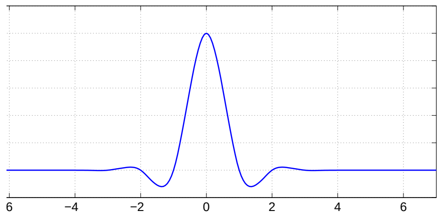
Figure 5.2: *The case* $m = 2$ *of Table 5.1.*



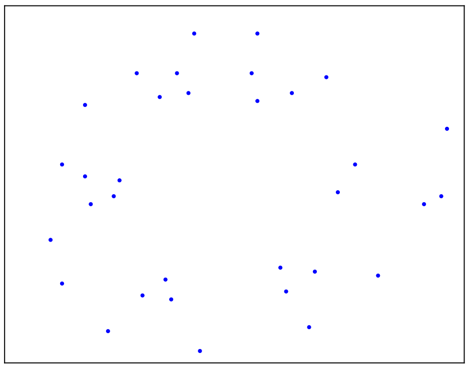
(a) $\phi_3^{\mathrm{I}}(-\frac{11}{2}|\cdot)$



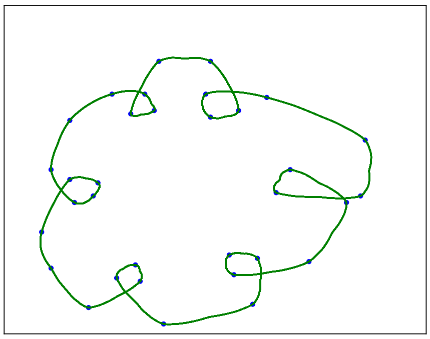
(b) $\phi_3^{\mathrm{I}}(-\frac{1}{2}|\cdot)$



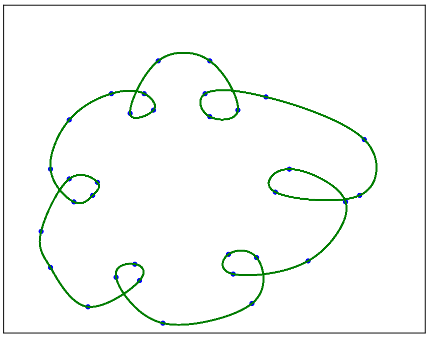
(c) $\phi_3^{\mathrm{I}}(\frac{1}{2}|\cdot)$



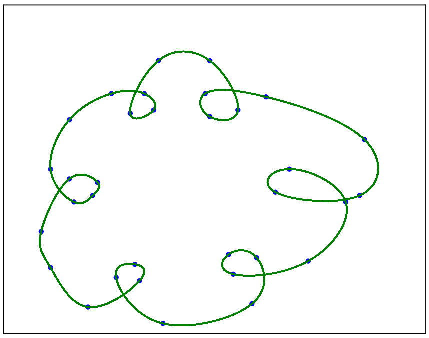
(d) Control points



(e) $\mathbf{\Phi}_3^{\mathrm{I}}(-\frac{11}{2}|\cdot)$



(f) $\mathbf{\Phi}_3^{\mathrm{I}}(-\frac{1}{2}|\cdot)$



(g) $\mathbf{\Phi}_3^{\mathrm{I}}(\frac{1}{2}|\cdot)$

Figure 5.3: *The case* $m = 3$ *of Table 5.1.*

5.2 Convergence based on Theorem 4.7.

In this section we proceed to show that, by applying the interpolatory refinable function existence and subdivision convergence result of our Theorem 4.7, an improvement on the results for $m = 1, 2, 3$ of Section 5.1, as based on Theorem 4.2, is obtained, in the sense that the t−intervals for convergence derived in this section will be strictly larger than the t−intervals in (5.11).

In order to apply Theorem 4.7, we first define, as in (2.35) of Definition 2.3.1, and for any $t \in \mathbb{R}$, the residual Laurent polynomial sequence $\left\{B_\ell^{I,m}(t|\cdot) : \ell = 1, \ldots, m\right\}$ by

$$B_\ell^{I,m}(t|z) := \frac{A_m^I(t|z)}{(1+z)^{2\ell}}, \quad z \in \mathbb{C} \setminus \{0\}, \quad \ell = 1, \ldots, m, \tag{5.15}$$

according to which then, as in (2.36),

$$A_m^I(t|z) = (1+z)^{2\ell} B_\ell^{I,m}(t|z), \quad z \in \mathbb{C} \setminus \{0\}, \quad \ell = 1, \ldots, m. \tag{5.16}$$

Also, observe from (5.16) and (5.5) that

$$B_\ell^{I,m}(t|z) = (1+z)^{2m-2\ell} B_m^I(t|z), \quad z \in \mathbb{C} \setminus \{0\}, \quad \ell = 1, \ldots, m, \tag{5.17}$$

for any $t \in \mathbb{R}$, and thus

$$B_m^{I,m}(t|z) = B_m^I(t|z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{5.18}$$

Next, for $\ell = 1, \ldots, m$, we define the sequence $b^{I,m,\ell}(t) = \{b_j^{I,m,\ell}(t)\} \in M_0(\mathbb{Z})$ by

$$\sum_j b_j^{I,m,\ell}(t) z^j := B_\ell^{I,m}(t|z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{5.19}$$

Since $A_m^I(t|\cdot) \in \mathcal{A}_{m,m+1}$, if $t \neq 0$, it follows from (1.23) in condition (b) of Theorem 1.2, together with (5.16) and (5.19), that, for any $t \in \mathbb{R} \setminus \{0\}$,

$$\operatorname{supp}\left\{b_j^{I,m,\ell}(t)\right\} = [-2m-1, 2m+1-2\ell]|_{\mathbb{Z}}\,. \tag{5.20}$$

We proceed to apply our existence and convergence result of Theorem 4.7 to the cases $m = 1, 2, 3$.

According to Theorem 4.7, we shall now seek to obtain, for $m = 1, 2, 3$, and for any given $\ell \in \{1, \ldots, m\}$, the t−intervals for which

$$\left\|S_{\mathbf{b}^{I,m,\ell}(t)}\right\|_\infty < 1, \tag{5.21}$$

or equivalently, from (2.5) in Theorem 2.1,

$$\rho_{m,\ell}(t) := \rho_{\mathbf{b}^{I,m,\ell}(t)} = max\Big\{ \sum_j \left|b_{2j}^{I,m,\ell}(t)\right|, \sum_j \left|b_{2j+1}^{I,m,\ell}(t)\right| \Big\} < 1. \tag{5.22}$$

To simplify the notation, we define, for any $m \in \mathbb{N}$, $\ell \in \{1, \ldots, m\}$ and $t \in \mathbb{R}$,

$$\left.\begin{array}{rcl} \alpha_{m,\ell}(t) & := & \sum_j \left|b_{2j}^{I,m,\ell}(t)\right|; \\ \beta_{m,\ell}(t) & := & \sum_j \left|b_{2j+1}^{I,m,\ell}(t)\right|. \end{array}\right\} \tag{5.23}$$

Hence we shall seek to obtain, for $m = 1, 2, 3$, and any given $\ell \in \{1, \ldots, m\}$, the t−interval for which it holds that

$$\rho_{m,\ell}(t) := max\Big\{\alpha_{m,\ell}(t), \beta_{m,\ell}(t)\Big\} < 1. \tag{5.24}$$

After noting from (5.19), (5.17) and (5.6) that, for $\ell \in \{1, \ldots, m\}$ and $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} \sum_j b_j^{I,m,\ell}(t) z^j &= \frac{1}{2^{2m+1}} z^{-m-1}(1+z)^{2m-2\ell} \Big\{ 4(1-t)z \sum_{j=0}^{m-1} \binom{m+j-1}{j} \Big[\tfrac{1}{2}\big(1 - \frac{z+z^{-1}}{2}\big)\Big]^j \\ &\quad + t(1+z)^2 \sum_{j=0}^{m} \binom{m+j}{j} \Big[\tfrac{1}{2}\big(1 - \frac{z+z^{-1}}{2}\big)\Big]^j \Big\}, \end{aligned} \tag{5.25}$$

we proceed to compute Table 5.2 by setting $m = 1, 2, 3$, and $\ell = 1, \ldots, m$, in the formula (5.25).

By using Table 5.2, we obtain the t-intervals, as given in Table 5.3, for which respectively the sums $\alpha_{m,\ell}(t)$ and $\beta_{m,\ell}(t)$, as given by (5.23), are less than one.

Based on Table 5.3, we therefore have the t−intervals, as given in Table 5.4, for which $\rho_{m,\ell}(t)$, as defined in (5.24), is less than one.

Hence we may now apply Theorem 4.7, together with Table 5.4, to deduce that we have the following t-intervals on which interpolatory refinable function existence and subdivision convergence with respect to the sequence $\left\{a_j^{I,m}(t)\right\}$, for respectively $m = 1, 2$, or 3, are guaranteed.

$$\left.\begin{aligned} m = 1: &\qquad t \in \Big(-2, 4\Big), \\ m = 2: &\qquad t \in \Big(-\tfrac{32}{3}, 8\Big), \\ m = 3: &\qquad t \in \Big(-\tfrac{280}{15}, \tfrac{252}{15}\Big). \end{aligned}\right\} \tag{5.26}$$

Observe that the t—intervals in (5.26) are substantially larger than the ones in (5.11),

Table 5.2: *The coefficient sequences* $\{b_{2j}^{I,m,\ell}(t)\}$ *and* $\{b_{2j+1}^{I,m,\ell}(t)\}$ *on their supports.*

m	ℓ	$\left\{b_{2j}^{I,m,\ell}(t)\right\}_{j=-m-1}^{j=m-\ell}$
1	1	$\frac{1}{16}\Big\{2t, 2t\Big\}$
2	1	$\frac{1}{256}\Big\{-6t, 32+6t, -6t\Big\}$
2	2	$\frac{1}{256}\Big\{-12t, 64-24t, -12t\Big\}$
3	1	$\frac{1}{2048}\Big\{10t, -48-30t, 304+20t, 304+20t, -48-30t, 10t\Big\}$
3	2	$\frac{1}{2048}\Big\{20t, -96, 320-40t, -96, 20t\Big\}$
3	3	$\frac{1}{2048}\Big\{30t, -144+30t, -144+130t, 30t\Big\}$
m	ℓ	$\left\{b_{2j+1}^{I,m,\ell}(t)\right\}_{j=-m-1}^{j=m-\ell}$
1	1	$\frac{1}{16}\Big\{-t, 8-2t, -t\Big\}$
2	1	$\frac{1}{256}\Big\{3t, -16, 96-6t, -16, 3t\Big\}$
2	2	$\frac{1}{256}\Big\{3t, -16+21t, -16+21t, 3t\Big\}$
3	1	$\frac{1}{2048}\Big\{-5t, 24+10t, -128+5t, 720-20t, -128+5t, 24+10t, -5t\Big\}$
3	2	$\frac{1}{2048}\Big\{-5t, 24-25t, 40+30t, 40+30t, 24-25t, -5t\Big\}$
3	3	$\frac{1}{2048}\Big\{-5t, 24-80t, 304-150t, 24-80t, -5t\Big\}$

Table 5.3: *The* t*-intervals on which respectively* $\alpha_{m,\ell}(t) < 1$ *and* $\beta_{m,\ell}(t) < 1$.

m	ℓ	t-intervals on which $\alpha_{m,\ell}(t) < 1$	t-intervals on which $\beta_{m,\ell}(t) < 1$
1	1	$-4 < t < 4$	$-2 < t < 4$
2	1	$-\frac{40}{3} < t < \frac{24}{3}$	$-\frac{32}{3} < t < \frac{80}{3}$
2	2	$-\frac{12}{3} < t < \frac{20}{3}$	$-\frac{14}{3} < t < \frac{18}{3}$
3	1	$-\frac{344}{15} < t < \frac{168}{15}$	$-\frac{280}{15} < t < \frac{692}{15}$
3	2	$-\frac{288}{15} < t < \frac{408}{15}$	$-\frac{260}{15} < t < \frac{252}{15}$
3	3	$-\frac{55}{10} < t < \frac{73}{10}$	$-\frac{53}{10} < t < \frac{75}{10}$

Table 5.4: *The* t*-intervals on which* $\rho_{m,\ell}(t) < 1$.

m	ℓ	t-interval on which $\rho_{m,\ell} < 1$.
1	1	$-2 < t < 4$
2	1	$-\frac{32}{3} < t < 8$
2	2	$-\frac{12}{3} < t < \frac{18}{3}$
3	1	$-\frac{280}{15} < t < \frac{168}{15}$
3	2	$-\frac{260}{15} < t < \frac{252}{15}$
3	3	$-\frac{53}{10} < t < \frac{73}{10}$

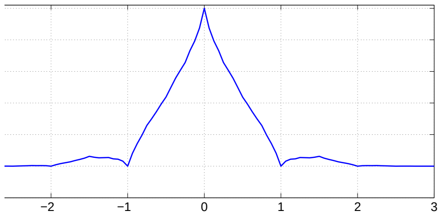
Table 5.5: *The sequences* $\{a_j^{I,m}(t)\}$ *on their supports for* $m = 1, 2, 3$ *and with* t *inside the intervals (5.26).*

m	t	$\left\{a_j^{I,m}(t)\right\}_{j=-2m-1}^{j=2m+1}$
1	-1	$\frac{1}{16}\left\{1,0,7,16,7,0,1\right\}$
1	$\frac{3}{2}$	$\frac{1}{32}\left\{-3,0,19,32,19,0,-3\right\}$
1	$\frac{7}{2}$	$\frac{1}{32}\left\{-7,0,23,32,23,0,-7\right\}$
2	-10	$\frac{1}{128}\left\{-15,0,37,0,42,128,42,0,37,0,-15\right\}$
2	$\frac{3}{2}$	$\frac{1}{512}\left\{9,0,-59,0,306,512,306,0,-59,0,9\right\}$
2	7	$\frac{1}{256}\left\{21,0,-79,0,186,256,186,0,-79,0,21\right\}$
3	-18	$\frac{1}{1024}\left\{45,0,-213,0,305,0,375,1024,375,0,305,0,-213,0,45\right\}$
3	$\frac{3}{2}$	$\frac{1}{4096}\left\{-15,0,123,0,-535,0,2475,4096,2475,0,-535,0,123,0,-15\right\}$
3	16	$\frac{1}{256}\left\{-10,0,53,0,-115,0,200,256,200,0,-115,0,53,0,-10\right\}$

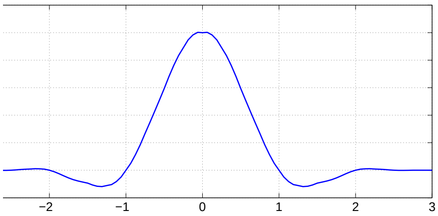
which justifies our statement in the introductory paragraph of this chapter to the effect that, for the cases $m = 1, 2, 3$ of the one-parameter family Laurent polynomials $A_m^I(t|\cdot)$ defined by (5.1) and (4.1), the interpolatory refinable function existence and subdivision convergence result of our Theorem 4.7 improves on the one based on Theorem 4.2.

For selected values of t inside the convergence intervals specified by (5.26), we now use the formulas (5.12),(5.13) and (5.14), together with (5.3), to obtain Table 5.5.

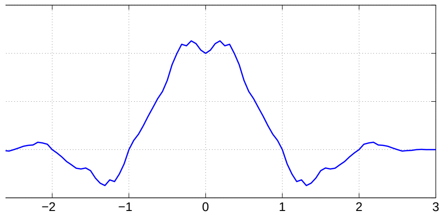
Graphical illustrations are provided in Figures 5.4 - 5.6 of the interpolatory refinable function $\phi_m^I(t|\cdot)$, and the corresponding limit subdivision curves $\mathbf{\Phi}_{m,\mathbf{c}}^I(t|\cdot)$, for control point sequences $\mathbf{c} = \{\mathbf{c}_j\}$ as shown.



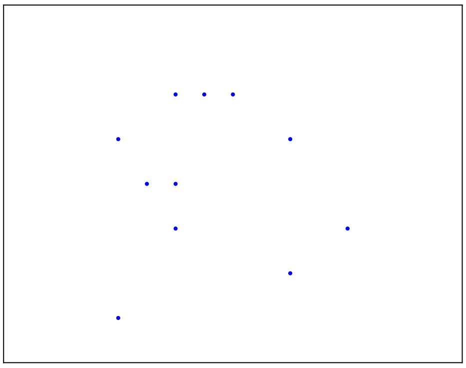
(a) $\phi_1^I(-1|\cdot)$



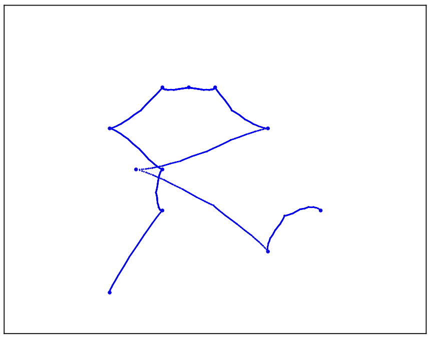
(b) $\phi_1^I(\frac{3}{2}|\cdot)$



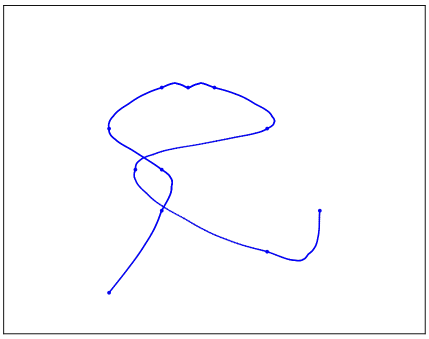
(c) $\phi_1^I(\frac{7}{2}|\cdot)$



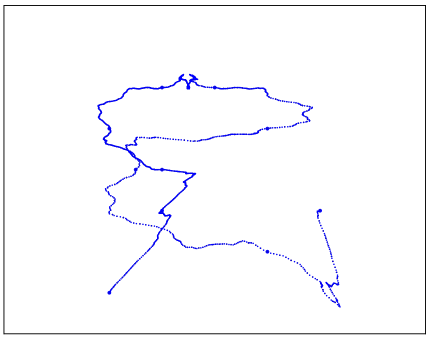
(d) Control points



(e) $\mathbf{\Phi}_1^I(-1|\cdot)$

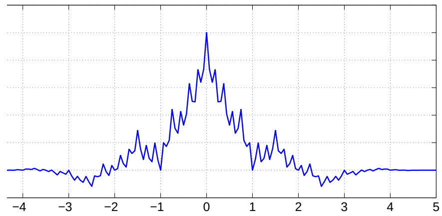


(f) $\mathbf{\Phi}_1^I(\frac{3}{2}|\cdot)$

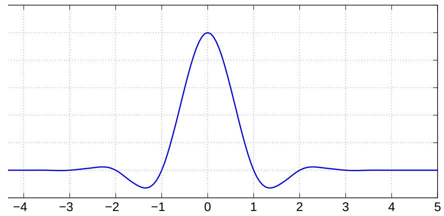


(g) $\mathbf{\Phi}_1^I(\frac{7}{2}|\cdot)$

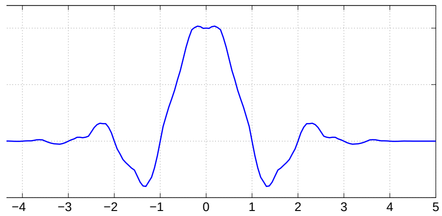
Figure 5.4: *The case* $m = 1$ *of Table 5.5.*



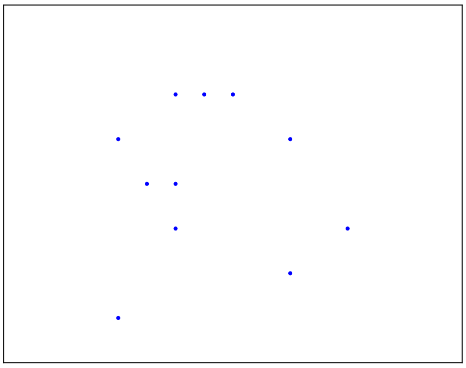
(a) $\phi_2^I(-10|\cdot)$



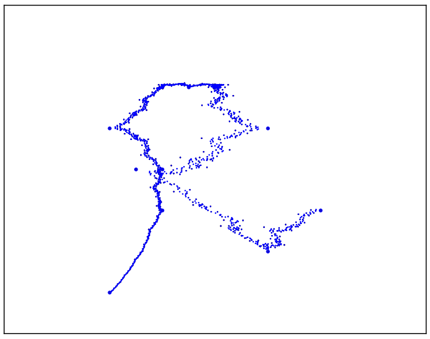
(b) $\phi_2^I(\frac{3}{2}|\cdot)$



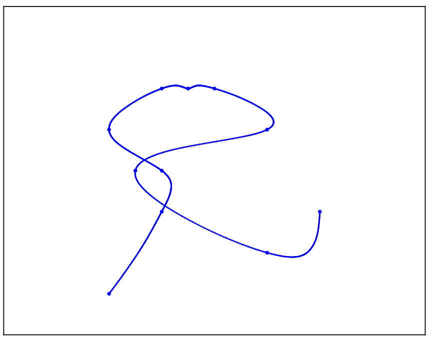
(c) $\phi_2^I(7|\cdot)$



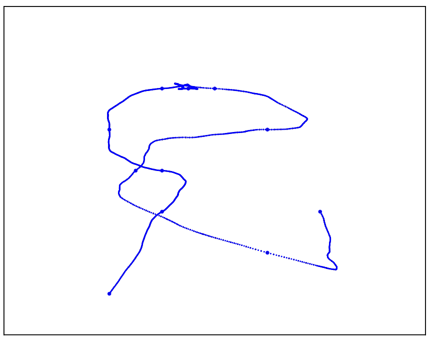
(d) Control points



(e) $\mathbf{\Phi}_2^I(-10|\cdot)$

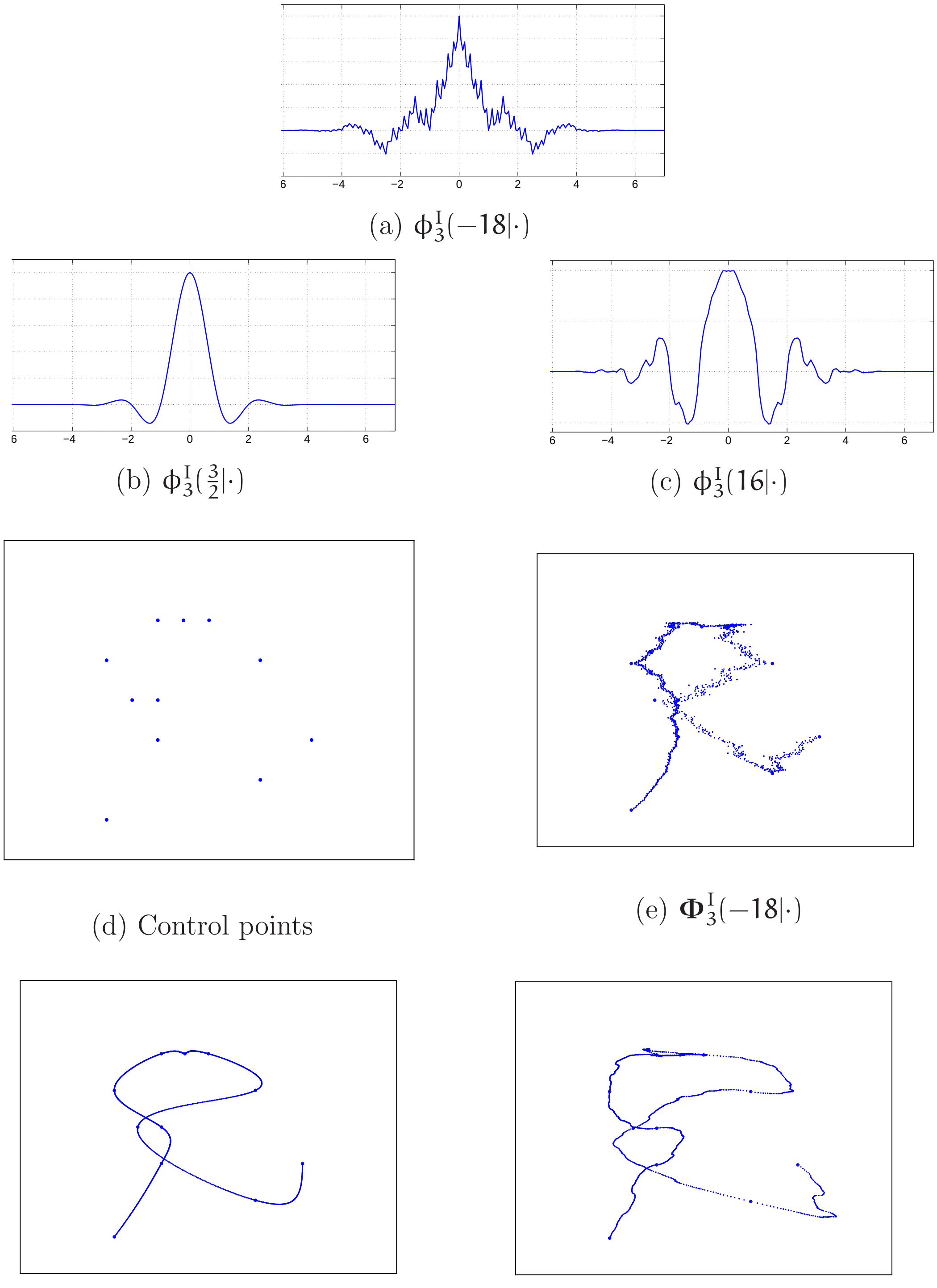


(f) $\mathbf{\Phi}_2^I(\frac{3}{2}|\cdot)$



(g) $\mathbf{\Phi}_2^I(7|\cdot)$

Figure 5.5: *The case* $m = 2$ *of Table 5.5.*



(a) $\phi_3^{\mathrm{I}}(-18|\cdot)$

(b) $\phi_3^{\mathrm{I}}(\frac{3}{2}|\cdot)$

(c) $\phi_3^{\mathrm{I}}(16|\cdot)$

(d) Control points

(e) $\mathbf{\Phi}_3^{\mathrm{I}}(-18|\cdot)$

(f) $\mathbf{\Phi}_3^{\mathrm{I}}(\frac{3}{2}|\cdot)$

(g) $\mathbf{\Phi}_3^{\mathrm{I}}(16|\cdot)$

Figure 5.6: *The case* $m = 3$ *of Table 5.5.*

Table 5.6: *The* $t-$*intervals for subdivision convergence.*

m	Our interval	[5]	[33]	[16]
1	$-2 < t < 4$	$-1 < t < 8$	$-4 < t < 4$	$-4 < t < 4$
2	$-\frac{800}{75} < t < \frac{600}{75}$	$-\frac{125}{75} < t < \frac{1000}{75}$	$-\frac{400}{75} < t < \frac{400}{75}$	$0 < t < \frac{128}{75}$
3	$-\frac{280}{15} < t < \frac{252}{15}$	$-\frac{33}{15} < t < \frac{264}{15}$		

The convergence of the one-parameter class of symmetric interpolatory subdivision schemes with mask sequence $\{a_j^{I,m}(t)\}$ defined by (5.3), (5.1) and (4.1), has also been studied, for specific (small) values of m, and by using alternative methods for analysis, in [5], [33] and [16]. In Table 5.6, we give, together with our convergence intervals (5.26) obtained by means of Theorem 4.7, the corresponding $t-$intervals for convergence established in, respectively, [5], [33] and [16].

Observe from Table 5.6 that, for $m = 2$ and $m = 3$, our result in (5.26) substantially improves on the results from [5], [33] and [16] for negative values of t.

5.3 Refinable function regularity

In this section, we present regularity (or smoothness) results for some of the interpolatory refinable functions $\phi = \phi_m^I(t|\cdot)$ for which graphs were drawn in this chapter. Observe from (5.9) that the limit subdivision curve $\mathbf{\Phi}_{m,\mathbf{c}}^I(t|\cdot)$ has the same regularity as $\phi_m^I(t|\cdot)$.

To this end, we first introduce the concept of Hölder regularity, as follows.

Definition 5.3.1 *For a function* $f : \mathbb{R} \longrightarrow \mathbb{R}$, *if there exist constants* $c \in [0, \infty)$ *and* $\alpha \in (0, 1]$ *such that*

$$|f(x) - f(y)| \leq c|x - y|^\alpha, \quad x, y \in \mathbb{R}, \tag{5.27}$$

then f *is said to be Hölder continuous on* $\mathbb{R}$, *with Hölder continuity exponent* α. *The class of all such functions is denoted by* $H^\alpha(\mathbb{R})$.

The Hölder continuity exponent $\alpha \in (0, 1]$ of a function $f \in H^\alpha(\mathbb{R})$ can be interpreted as a measure of the regularity of f, in the sense of the following embedding result from

[5, Lemma 6.3.1].

LEMMA 5.2 *For* $0 < \tilde{\alpha} \leq \alpha \leq 1$,

$$C_0^1(\mathbb{R}) \subset H_0^1(\mathbb{R}) \subset H_0^{\alpha}(\mathbb{R}) \subset H_0^{\tilde{\alpha}}(\mathbb{R}) \subset C_0(\mathbb{R}), \tag{5.28}$$

where $H_0^{\alpha}(\mathbb{R}) := H^{\alpha}(\mathbb{R}) \cap C_0(\mathbb{R})$.

DEFINITION 5.3.2 *For* $k = 0, 1, \dots$, *and* $\alpha \in (0, 1]$, *the function space*

$$C^{k,\alpha}(\mathbb{R}) := \{f \in C^k(\mathbb{R}) : \; f^{(k)} \in H^{\alpha}(\mathbb{R})\} \tag{5.29}$$

is called the Hölder continuity space of order k *with Hölder continuity exponent* α. *Also, define*

$$C_0^{k,\alpha}(\mathbb{R}) := C^{k,\alpha}(\mathbb{R}) \cap C_0(\mathbb{R}), \tag{5.30}$$

and observe that $C^{0,\alpha}(\mathbb{R}) = H^{\alpha}(\mathbb{R})$ *and* $C_0^{0,\alpha}(\mathbb{R}) = H_0^{\alpha}(\mathbb{R})$.

The following existence and regularity result is from [5, Theorem 8.4.1].

THEOREM 5.3 *For an arbitrary integer* $m \in \mathbb{N}$, *let* $t \in \mathbb{R}$ *denote any parameter for which the condition*

$$1 - \frac{2^{2m-1}}{\binom{2m-1}{m-1}} < t < 8\left[\frac{2^{2m-1}}{\binom{2m-1}{m-1}} - 1\right] \tag{5.31}$$

is satisfied. Then there exists an interpolatory refinable function $\phi_m^I(t|\cdot)$ *with refinement sequence* $\{a_j^{I,m}(t)\}$, *as given by (5.3), (5.1) and (4.1), with*

$$\phi_m^I(t|\cdot) \in C_0^{\ell(t),\alpha(t)}(\mathbb{R}), \tag{5.32}$$

where

$$\ell(t) := 2m - 2 - \lfloor \sigma(t) \rfloor, \tag{5.33}$$

and for all

$$\alpha(t) \in (0, 1 + \lfloor \sigma(t) \rfloor - \sigma(t)), \tag{5.34}$$

with

$$\sigma(t) := \begin{cases} \log_2[(1-t)\binom{2m-1}{m-1}], & t \in (-\infty, 0); \\ \\ \log_2[(1+\frac{t}{8})\binom{2m-1}{m-1}], & t \in (0,\infty) \setminus \{1\}. \end{cases} \tag{5.35}$$

Observe that, by setting $m = 1, 2, 3$ in (5.31), we obtain precisely the column under [5] in Table 5.6

By using Theorem 5.3, we may therefore obtain the regularity results given in Table 5.7 for those interpolatory refinable function $\phi_m(t|\cdot)$ drawn in Figures 5.4 - 5.6 with $t-$values inside the interval implied by (5.31).

The graphs in this chapter suggest that the smoothness of $\phi_m^I(t|\cdot)$ decreases significantly (to almost fractal-like nature in some instances) as t approaches the endpoints of the convergence interval. This observation is supported to some extent by the regularity results of Table 5.7.

However, for example, the regularity of $\phi_3^I(-18|\cdot)$ in Figure 5.6 (a) $\phi_3^I(-18|\cdot)$ is not covered by Theorem 5.3, since $-18 \notin \left(-\frac{33}{15}, \frac{264}{15}\right)$, and is therefore not included in Table 5.7. Hence there exists a need for regularity results for all interpolatory refinable function obtained from Theorem 4.7. We intend to pursue this issue in further research.

Table 5.7: *The regularity (or smoothness) of the interpolatory refinable function* $\phi^I_m(t|\cdot)$, *for* t *inside the interval implied by* (5.31).

m	t	Figure	regularity (or smoothness) of $\phi^I_m(t\lvert\cdot)$
1	$-\frac{1}{2}$	5.1 (b)	$\phi^I_1(-\frac{1}{2}\lvert\cdot) \in C_0^{0,\alpha(-\frac{1}{2})}(\mathbb{R}) = H_0^{\alpha(-\frac{1}{2})}(\mathbb{R})$, for all $\alpha(-\frac{1}{2}) \in (0, 1-\log_2(\frac{3}{2})) \approx (0, 0.4150)$
1	$\frac{1}{2}$	5.1 (c)	$\phi^I_1(\frac{1}{2}\lvert\cdot) \in C_0^{0,\alpha(\frac{1}{2})}(\mathbb{R}) = H_0^{\alpha(\frac{1}{2})}(\mathbb{R})$, for all $\alpha(\frac{1}{2}) \in (0, 1-\log_2(\frac{17}{16})) \approx (0, 0.9125)$
1	$\frac{3}{2}$	5.4 (b)	$\phi^I_1(\frac{3}{2}\lvert\cdot) \in C_0^{0,\alpha(\frac{3}{2})}(\mathbb{R}) = H_0^{\alpha(\frac{3}{2})}(\mathbb{R})$, for all $\alpha(\frac{3}{2}) \in (0, 1-\log_2(\frac{19}{16})) \approx (0, 0.7520)$
1	$\frac{7}{2}$	5.4 (c)	$\phi^I_1(\frac{7}{2}\lvert\cdot) \in C_0^{0,\alpha(\frac{7}{2})}(\mathbb{R}) = H_0^{\alpha(\frac{7}{2})}(\mathbb{R})$, for all $\alpha(\frac{7}{2}) \in (0, 1-\log_2(\frac{23}{16})) \approx (0, 0.4764)$
2	$-\frac{1}{2}$	5.2 (b)	$\phi^I_2(-\frac{1}{2}\lvert\cdot) \in C_0^{0,\alpha(-\frac{1}{2})}(\mathbb{R}) = H_0^{\alpha(-\frac{1}{2})}(\mathbb{R})$, for all $\alpha(-\frac{1}{2}) \in (0, 3-\log_2(\frac{9}{2})) \approx (0, 0.830)$
2	$\frac{1}{2}$	5.2 (c)	$\phi^I_2(\frac{1}{2}\lvert\cdot) \in C_0^{1,\alpha(\frac{1}{2})}(\mathbb{R})$, for all $\alpha(\frac{1}{2}) \in (0, 2-\log_2(\frac{51}{16})) \approx (0, 0.3275)$
2	$\frac{3}{2}$	5.5 (b)	$\phi^I_2(\frac{3}{2}\lvert\cdot) \in C_0^{1,\alpha(\frac{3}{2})}(\mathbb{R})$, for all $\alpha(\frac{3}{2}) \in (0, 2-\log_2(\frac{57}{16})) \approx (0, 0.1671)$
2	7	5.5 (c)	$\phi^I_2(7\lvert\cdot) \in C_0^{0,\alpha(7)}(\mathbb{R}) = H_0^{\alpha(7)}(\mathbb{R})$, for all $\alpha(7) \in (0, 3-\log_2(\frac{45}{8})) \approx (0, 0.5081)$
3	$-\frac{1}{2}$	5.3 (b)	$\phi^I_3(-\frac{1}{2}\lvert\cdot) \in C_0^{1,\alpha(-\frac{1}{2})}(\mathbb{R})$, for all $\alpha(-\frac{1}{2}) \in (0, 4-\log_2(15)) \approx (0, 0.0931)$
3	$\frac{1}{2}$	5.3 (c)	$\phi^I_3(\frac{1}{2}\lvert\cdot) \in C_0^{1,\alpha(\frac{1}{2})}(\mathbb{R})$, for all $\alpha(\frac{1}{2}) \in (0, 4-\log_2(\frac{170}{16})) \approx (0, 0.5906)$
3	$\frac{3}{2}$	5.6 (b)	$\phi^I_3(\frac{3}{2}\lvert\cdot) \in C_0^{1,\alpha(\frac{3}{2})}(\mathbb{R})$, for all $\alpha(\frac{3}{2}) \in (0, 4-\log_2(\frac{190}{16})) \approx (0, 0.4301)$
3	16	5.6 (c)	$\phi^I_3(16\lvert\cdot) \in C_0^{0,\alpha(16)}(\mathbb{R}) = H_0^{\alpha(16)}(\mathbb{R})$, for all $\alpha(16) \in (0, 5-\log_2(30)) \approx (0, 0.0931)$

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